

Pacific Journal of Mathematics

**ON A CHARACTERIZATION OF INFINITE COMPLEX
MATRICES MAPPING THE SPACE OF ANALYTIC SEQUENCES
INTO ITSELF**

LOUISE ARAKELIAN RAPHAEL

ON A CHARACTERIZATION OF INFINITE COMPLEX MATRICES MAPPING THE SPACE OF ANALYTIC SEQUENCES INTO ITSELF

LOUISE A. RAPHAEL

Let S be the space of all complex sequences. An element $u = \{u_n\}_{n=0}^\infty$ of S is called analytic if for some constant $M > 0$, $|u_n| \leq M^{n+1}$ for $n = 0, 1, 2, \dots$. By A denote the space of all analytic sequences. Clearly A is the space of all complex functions analytic at zero. I. Heller has proved

Theorem 1. The transformation $y_n = \sum_{m=0}^\infty c_{nm}u_m$ maps A into A if and only if for every $p > 0$ there exists a $q > 0$ and a constant $M > 0$ such that $|c_{nm}| \leq Mp^m/q^n$ for $m, n = 0, 1, 2, \dots$; and also if and only if the function G of two complex variables (i.e., in $E \times E$, where E is the complex plane) represented by the double power series $G(z, y) = \sum_{m, n=0}^\infty c_{nm}z^m y^n$ be regular on $E \times 0$.

The present paper provides an alternative proof for the theorem in order to give insight into the structure of A as a countable union of BK spaces, that is, Banach spaces with continuous coordinates.

Let $q > 0$ be fixed and $A_q = \{u \in S \mid \sup_n |q^n u_n| = \|u\|_q < \infty, n = 0, 1, 2, \dots\}$.

THEOREM 2. (1) $A = \bigcup_{n=0}^\infty A_{q_n}$ where $q_n \downarrow 0$, and
 (2) for any $q > 0$, $(A_q, \|u\|_q)$ is a BK space.

Proof. (1) A complex sequence $u = \{u_n\}_{n=0}^\infty$ is analytic if and only if the $\sup_n |q^n u_n| \leq M$ for some $q > 0$, some constant $M > 0$ and $n = 0, 1, 2, \dots$. It now follows that $A = \bigcup_{0 < q < \infty} A_q$. The proof is completed by a set theoretic argument showing that $\bigcup_{0 < q < \infty} A_q = \bigcup_{n=0}^\infty A_{q_n}$ after observing that if $0 < r < s$, then $A_s \subset A_r$.

(2) It suffices to observe that $(A_q, \|u\|_q)$ is isometrically isomorphic with the Banach space of all bounded complex sequences

$$(m) = \{u \in S \mid \|u\|_{(m)} = \sup_n |u_n|\}.$$

The operator E_q from A_q into (m) establishing this isomorphism is defined by $E_q: \{u_n\}_{n=0}^\infty \rightarrow \{q^n u_n\}_{n=0}^\infty$. Finally for each n , $|u_n| \leq \|u\|_q/q^n$. Thus the coordinate functional $P_n(u) = u_n$ is continuous, being a linear operator on A_q . This proves that the space $(A_q, \|u\|_q)$ is a BK space.

By a mapping C of a sequence space X into a sequence space Y generated by an infinite complex matrix (c_{nm}) , $m, n = 0, 1, 2, \dots$ is

meant $(y = C(u), u \in X)$ if and only if $(y_n = \sum_{m=0}^{\infty} c_{nm}u_m, y = \{y_n\}_{n=0}^{\infty} \in Y)$.

THEOREM 3. *Let C be the transformation from A into A generated by an infinite complex matrix (c_{nm}) $n, m = 0, 1, 2, \dots$. For each $p > 0$ and $q > 0$ fixed let $A_{pq} = \{u \in A_p \mid C(u) \in A_q\}$. Then*

- (1) $A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$ where $q_n \downarrow 0$, and
- (2) for each $p > 0$ and $q > 0$ fixed,

$$(A_{pq}, \|u\|_{pq} = \|u\|_p + \|C(u)\|_q)$$

is a *BK space*.

Proof. (1)

$$\begin{aligned} A_p &= \left\{ u \in A_p \mid C(u) \in A = \bigcup_{n=0}^{\infty} A_{q_n}, q_n \downarrow 0 \right\} \\ &= \bigcup_{n=0}^{\infty} \{u \in A_p \mid C(u) \in A_{q_n}\} = \bigcup_{n=0}^{\infty} A_{pq_n}. \end{aligned}$$

(2) For each $u = \{u_n\}_{n=0}^{\infty}$ belonging to the *BK space* A_p , $(C(u))_k = C_k(u) = \sum_{n=0}^{\infty} c_{kn}u_n$ on A_p is the limit of the sequence of continuous linear operators $\sum_{n=0}^j c_{kn}u_n$ $j = 0, 1, 2, \dots$ on A_p . So C_k is a continuous linear operator on A_p for each $k = 0, 1, 2, \dots$ by [2, Th. 17, p. 54]. This shows that C is a continuous linear operator from A_p into A .

The *BK spaces* $(A_p, \|u\|_p)$, $(A_q, \|u\|_q)$ and the continuous linear map $C : A_p \rightarrow A$ satisfy the conditions of [4, Th. 1, p. 226]. This together with [4, Th. 3, p. 205] prove that $A_p \cap C^{-1}(A_q) = A_{pq}$ is a *FK space* (Frechet space with continuous coordinates) with the norm $\|u\|_p + \|C(u)\|_q$ (as the sup of two normed topologies is given by the sum of the norms). That $(A_{pq}, \|u\|_{pq})$ is a *BK space* is now immediate.

THEOREM 4. *Let C be the transformation from A into A generated by an infinite complex matrix (c_{nm}) $n, m = 0, 1, 2, \dots$. Then*

- (1) for every $p > 0$ there exists a $q > 0$ such that C maps A_p into A_q .

The transformation C from A_p into A_q generated by (c_{nm}) for fixed $p > 0$ and $q > 0$

- (2) *is linear and continuous, and*
- (3) *its norm, $\|C\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m}, n = 0, 1, 2, \dots$.*

Proof. (1) For any $p > 0, C : A_p \rightarrow A = \bigcup_{n=0}^{\infty} A_{q_n}, q_n \downarrow 0$. Moreover $A_p = \bigcup_{n=0}^{\infty} A_{pq_n}$. And by definition of the Banach norm $\|u\|_{pq} = \|u\|_p + \|C(u)\|_q$ on A_{pq} , the injective maps from A_{pq_n} into A_p are

continuous for any $p > 0$. Thus by [4, Corollary 6, p. 205] or [5, Satz 4.6, p. 472], there exists an index k such that $A_p = A_{pq_k}$. This q_k is the desired q .

(2) The linearity of C is clear. Continuity follows from [4, Corollary 5, p. 204].

(3) Map A_p into (m) by the operator $E_p: u = \{u_m\}_{m=0}^{\infty} \rightarrow \{p^m u_m\}_{m=0}^{\infty}$. Define the operator B to be $E_q C E_p^{-1}$. Clearly B is an operator from (m) into (m) which is generated by the infinite matrix

$$(b_{nm}) = (q^n c_{nm} p^{-m}).$$

And so B is linear and continuous from (m) into (m) and $\|B\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m}$ $n = 0, 1, 2, \dots$. But $\|C\| = \|B\|$.

Proof of Theorem 1. By Theorem 4 (1) and (3) for every $p > 0$ there exists a $q > 0$ such that C maps A_p into A_q and

$$C\| = \sup_n \sum_{m=0}^{\infty} q^n |c_{nm}| p^{-m} \leq M, n = 0, 1, 2, \dots$$

respectively. Thus $|c_{nm}| \leq Mp^m/q^n$, $m, n = 0, 1, 2, \dots$. This proves necessity.

Since $A = \bigcup_{0 < p < \infty} A_p$, it suffices to show that the operator C is well defined on A_p . Let $0 < r < 1$. For the number pr there exists a number $q > 0$ such that $|c_{nm}| \leq M(pr)^m q^{-n}$ for all m and n and some M , and so $|c_{nm} u_m| \leq Mr^m q^{-n} \|u\|_p$ for all m and n . This implies that the series $\sum_{m=0}^{\infty} c_{nm} u_m$ is convergent and

$$\left| \sum_{m=0}^{\infty} c_{nm} u_m \right| \leq M(1-r)^{-1} q^{-n} \|u\|_p.$$

Thus the sequence $y = C(u)$ belongs to the space A_q and therefore also to the space A . This proves the sufficiency of the condition.

The functional analysis method employed herein has implications beyond the proof of Theorem 1. It enables us to extend Heller's result to the space of Borel measurable functions bounded with respect to a weight function. This will be the subject of a forthcoming paper.

It is a pleasure to thank Professors W. Bogdanowicz, I. Heller and the referee for their critical readings and valuable suggestions.

BIBLIOGRAPHY

1. S. Banach, *Theorie des operation lineaires*, Monografje Matematyczne, Warsaw, 1932.
2. N. Dunford and J. Schwartz, *linear operators*, I, Interscience, New York, 1964.

3. I. Heller, *Contributions to the theory of divergent series*, Pacific J. Math **2** (1952), 153-177.
4. A. Wilansky, *Functional analysis*, Blaisdell, New York, 1964.
5. K. Zeller, *Allgemeine eigenschaften von limiterungsverfahren*, Math. Zeitsch. **53** (1951), 463-487.

Received August 11, 1967. Research was sponsored by the Office of Naval Research (Nonr-4311(00)).

CATHOLIC UNIVERSITY AND
HOWARD UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

R. R. PHELPS

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Willard Ellis Baxter, <i>On rings with proper involution</i>	1
Donald John Charles Bures, <i>Tensor products of W^*-algebras</i>	13
James Calvert, <i>Integral inequalities involving second order derivatives</i>	39
Edward Dewey Davis, <i>Further remarks on ideals of the principal class</i>	49
Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients I</i>	53
Francis James Flanigan, <i>Algebraic geography: Varieties of structure constants</i>	71
Denis Ragan Floyd, <i>On QF – 1 algebras</i>	81
David Scott Geiger, <i>Closed systems of functions and predicates</i>	95
Delma Joseph Hebert, Jr. and Howard E. Lacey, <i>On supports of regular Borel measures</i>	101
Martin Edward Price, <i>On the variation of the Bernstein polynomials of a function of unbounded variation</i>	119
Louise Arakelian Raphael, <i>On a characterization of infinite complex matrices mapping the space of analytic sequences into itself</i>	123
Louis Jackson Ratliff, Jr., <i>A characterization of analytically unramified semi-local rings and applications</i>	127
S. A. E. Sherif, <i>A Tauberian relation between the Borel and the Lototsky transforms of series</i>	145
Robert C. Sine, <i>Geometric theory of a single Markov operator</i>	155
Armond E. Spencer, <i>Maximal nonnormal chains in finite groups</i>	167
Li Pi Su, <i>Algebraic properties of certain rings of continuous functions</i>	175
G. P. Szegő, <i>A theorem of Rolle's type in E^n for functions of the class C^1</i>	193
Giovanni Viglino, <i>A co-topological application to minimal spaces</i>	197
B. R. Wenner, <i>Dimension on boundaries of ε-spheres</i>	201