A TAUBERIAN RELATION BETWEEN THE BOREL AND THE LOTOTSKY TRANSFORMS OF SERIES

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This paper is concerned with the equiconvergence of the Lototsky transform and the Borel (exponential) transform for a class of series satisfying the Tauberian condition \( a_n = o(1) \).

If \( s_n = a_0 + a_1 + \cdots + a_n \), the Borel (exponential) transform \( f(x) \) of \( s_n \) is usually defined by

\[
e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!}.
\]

Writing \( s_n = a_1 + a_2 + \cdots + a_n \), the Lototsky transform \( \sigma_n \) of \( s_n \) introduced by A. V. Lototsky [8] is defined by

\[
(1.1) \quad \sigma_n = \frac{1}{n!} \sum_{k=1}^{n} p_{n,k} s_k,
\]

where \( p_{n,k} \) is the coefficient of \( x^k \) in

\[
p_n(x) = x(x+1)(x+2) \cdots (x+n-1), \quad (n = 1, 2, \cdots).
\]

Thus it is usual in considering Lototsky summability to take the first term of the series as \( a_1 \), and in considering Borel summability\(^1\) to take it as \( a_0 \). In order to compare the methods without changing the customary notation we will therefore apply the Borel methods to the series \( 0 + a_1 + a_2 + \cdots \) and apply the Lototsky method to the series \( a_1 + a_2 + \cdots \). We recall (Hardy [5] pp. 182–3) that the Borel summability of \( a_1 + a_2 + \cdots \) implies the Borel summability \( 0 + a + a + \cdots \), but not conversely. The two methods are equivalent if (and only if) \( a_n \rightarrow 0(B) \); this is true in particular if

\[
(1.2) \quad a_n = o(1),
\]

and thus for the series considered in this paper.

Lototsky's transform is essentially a special case of a class of transformations introduced by J. Karamata [7]. It is the \((f, d_n)\) transform defined by G. Smith [11], when \( f(z) = z, \ d_n = n \), and the \([F, d_n]\) transform defined by A. Jakimorski [6], when \( d_n = n - 1 \) and \( n \geq 1 \). It is also the \( \sigma^\alpha \) method of summability introduced by Vučković [12], when \( \alpha = 1 \).

Numerous properties of this Lototsky transform and its relation

\(^1\) "Borel summability" is throughout taken to refer to Borel's exponential method.
with some of the other transformations have been shown in Agnew ([1], [3]).

In § 2 of the present paper we shall show that, for the class of series satisfying the Tauberian condition (1.2), the Lototsky transform $\sigma_n$ and the Borel transform $f(\log n)$ are equiconvergent. This includes the result that, under the condition (1.2), Lototsky summability implies Borel summability, and it should therefore be remarked that this result is essentially due to Agnew ([1], [3]). For we have, with Agnew’s notation, (since for suitably restricted sequences the starred and unstarred methods are equivalent)

$$L \subset BI^* \sim BI \sim B.$$ 

The argument of § 2 depends on an asymptotic expression for $p_{a,b}$ for large $n$ given by Moser and Wyman [10].

In § 3, we introduce a Tauberian constant for the Lototsky transform.

Agnew ([2] §’s 2, 3) has obtained a result of a similar nature to Theorem 3.1 of this paper but for the Borel transform instead.

We may observe that Theorem 3.1 is included in Theorem 2.1 of the present paper. Also, a “$O$” Tauberian theorem for the Lototsky transform is included in Theorem 2.1, but not in Theorem 3.1.

2. **Theorem 2.1.** Suppose that (1.2) holds. Then

$$\sigma_n - f(\log n) \to 0, \quad \text{as } n \to \infty.$$ 

For the proof of Theorem 2.1, we require the following lemmas.

**Lemma 2.1.** There is a $K = K(n)$ such that

$$p_{a_1} < p_{a_2} < \cdots < p_{a_K} \geq p_{a,K+1} > p_{a,K+2} > \cdots > p_{a,n}$$

and that for large $n$

$$K(n) = \log n + O(1).$$

The result is due to Hammersley [4]. Hammersley gives a more precise result than (2.2), but this is enough for our purposes.

**Lemma 2.2.** Let $a, b$ be constants with $0 < a < 1 < b$. Then for large $n$ uniformly in

$$a \log n \leq k \leq b \log n,$$

we have
(2.4) \[
\frac{P_{nk}}{n!} = O\left(\frac{1}{\sqrt{n \log n}} n^{\phi(\theta)}\right)
\]

where we write

(2.5) \[
\phi(\theta) = \theta - 1 - \theta \log \theta ; \quad \theta = \frac{k}{\log n}.
\]

Proof. Write

\[
f_n(t) = \sum_{\nu=0}^{n-1} \frac{t}{t+\nu}.
\]

We note that, for fixed \(n\), as \(t\) increases from 0 to \(\infty\), \(f_n(t)\) increases from 1 to \(n\).

Now, it follows from Moser and Wyman ([10], equation (4.51) and the line below it) that, uniformly in a bigger range which includes (2.3)

(2.6) \[
p_{nk} = \frac{\Gamma(n+R)}{(2\pi H)^{1/2} R^k \Gamma(R)} \left(1 + o\left(\frac{1}{H}\right)\right)
\]

where \(R\) is the unique positive solution of the equation

(2.7) \[
f_n(R) = k
\]

and where

(2.8) \[
H = k - \sum_{\nu=0}^{n-1} \frac{R^2}{(R+\nu)^2}.
\]

Now, it clearly follows from the definition that for large \(n\) uniformly in \(0 \leq t \leq c\) (\(c\) is a constant) we have

(2.9) \[
f_n(t) = t \log n + O(1).
\]

Choose \(c > b\); then it follows from (2.9) that, for sufficiently large \(n\)

\[f_n(c) > b \log n\]

and hence, for sufficiently large \(n\), we have \(R \leq C\) for all \(k\) satisfying (2.3).

In the rest of the proof of this lemma, the symbol \(O\) is to be taken as applying for large \(n\) uniformly for \(k\) in the range (2.3). Thus, by what has just been said, \(R = O(1)\). Also since (2.9) is valid for \(t = R\) we deduce from (2.7) that

(2.10) \[
R = \frac{k}{\log n} + O\left(\frac{1}{\log n}\right).
\]

We also note, that since \(R\) is bounded
(2.11) \( H = k + O(1) \).

Now, since \( R \) is bounded, it follows at once from Stirling’s approximation that

(2.12) \[ \frac{\Gamma(n + R)}{n!} = n^{R-1}\left(1 + O\left(\frac{1}{n}\right)\right). \]

However, if we consider \( \log(n^{R-1}) \) we find, by (2.10) that

(2.13) \[ \begin{align*}
\log(n^{R-1}) &= (R - 1) \log n = k - \log n + O(1) \\
&= (\theta - 1) \log n + O(1).
\end{align*} \]

Also, by (2.10)

(2.14) \[ \begin{align*}
\log(R^k) &= k \log R = k \log \theta + k \log\left(1 + O\left(\frac{1}{k}\right)\right) \\
&= (\theta \log \theta) \log n + O(1).
\end{align*} \]

Also, since \( R \geq K > 0 \), where \( K \) is a constant, we have

(2.15) \[ \frac{1}{\Gamma(R)} = O(1); \]
also by (2.11)

(2.16) \[ \frac{1}{\sqrt{2\pi H}} = O\left(\frac{1}{\sqrt{\log n}}\right). \]

Thus combining (2.6) and (2.12)-(2.16) the result (2.4) follows.

**Lemma 2.3.** Let \( \lambda \) be a constant so that

(2.17) \[ \frac{1}{2} < \lambda < \frac{2}{3}. \]

Then for large \( n \) uniformly in the range

(2.18) \[ |k - \log n| \leq (\log n)^3, \]

we have

(2.19) \[ \frac{p_{nk}}{n!} = \frac{1}{\sqrt{2\pi \log n}} \exp\left(-\frac{h^2}{2 \log n}\right) \times \left\{1 + O\left(\frac{|h| + 1}{\log n}\right) + \left(\frac{|h|^2}{\log^2 n}\right)\right\}, \]

where we write

(2.20) \[ k = \log n + h. \]
Proof. To prove (2.19) we need an improvement on (2.10). We have
\[ f_n(t) = \log n + \nu + O\left(\frac{1}{n}\right), \]
where \( \nu \) is Euler's constant. Hence by definition of \( R \)
\[ f_n(R) - f_n(1) = h - \nu + O\left(\frac{1}{n}\right). \]
But for some \( t \) between 1 and \( R \)
\[ f_n(R) - f_n(1) = (R - 1)f'_n(t). \]
Also for the relevant \( t \) we have, since \( R = O(1) \)
\[ f'_n(t) = \sum_{\nu=1}^{n-1} \frac{\nu}{(t+\nu)^2} = \sum_{\nu=1}^{n-1} \frac{1}{t+\nu} - t \sum_{\nu=1}^{n-1} \frac{1}{(t+\nu)^2} \]
\[ = \log n + O(1). \]
Thus
\[ h - \gamma + O\left(\frac{1}{n}\right) = (R - 1) \left( \log n + O(1) \right), \]
\[ R - 1 = \frac{(h - \gamma + O\left(\frac{1}{n}\right))}{\log n} \left( 1 + O\left(\frac{1}{\log n}\right) \right) \]
(2.21)
\[ = \frac{h - \gamma}{\log n} + O\left(\frac{|h| + 1}{\log^2 n}\right). \]
Since \( \Gamma(1) = 1 \) and since \( d/dt(1/\Gamma(t)) \) is bounded for \( t \) between 1 and \( R \), we have
(2.22)
\[ \frac{1}{\Gamma(R)} = 1 + O\left(\frac{|h| + 1}{\log n}\right). \]
Also
(2.23)
\[ \frac{1}{\sqrt{H}} = \frac{1}{\sqrt{k}} \left( 1 + O\left(\frac{1}{k}\right) \right); \]
(2.24)
\[ \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{\log n}} \left( 1 + O\left(\frac{|h|}{\log n}\right) \right). \]
Also
\[ \log n^{R-1} = (R - 1) \log n = h - \gamma + O\left(\frac{|h| + 1}{\log n}\right) \]
so that
(2.25)
\[ n^{R-1} = e^{h - \gamma} \left\{ 1 + O\left(\frac{|h| + 1}{\log n}\right) \right\}. \]
Up to this point, results are valid in the whole range (2.3) of Lemma 2.2, though they give an improvement on (2.3) when $|h| = o(\log n)$. But from now on, we take "O" as applying for large $n$ uniformly in $k$ in the range (2.18) only.

Consider $\log (R^k)$. We have

$$\log (R^k) = k \log R$$

$$= (\log n + h) \log \left(1 + \frac{h - \nu}{\log n} + O\left(\frac{|h| + 1}{\log^2 n}\right)\right)$$

$$= (\log n + h)\left(\frac{h - \nu}{\log n} - \frac{h^2}{2 \log^2 n} + O\left(\frac{|h| + 1}{\log^2 n}\right)\right)$$

$$+ O\left(\frac{|h|^3}{\log^3 n}\right)$$

$$= h - \gamma + \frac{h^2}{2 \log n} + O\left(\frac{|h| + 1}{\log n}\right) + O\left(\frac{|h|^3}{\log^3 n}\right).$$

Thus

$$R^k = \left\{ \exp \left(h - \gamma + \frac{h^2}{2 \log n}\right)\right\} \left(1 + O\left(\frac{|h| + 1}{\log n}\right)\right)$$

$$+ O\left(\frac{|h|^3}{\log^3 n}\right).$$

Combining (2.6), (2.12) and (2.22) – (2.26), the result (2.19) follows.

**Proof of Theorem 2.1.** Let $N$ be the integer nearest to $\log n$. Then we have, for $x = \log n$.

$$f(x) = e^{-x} \sum_{k=1}^{\infty} s_k \frac{x^k}{k!} = e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k!} (s_N + s_k - s_N)$$

$$= s_N + e^{-x} \sum_{k=1}^{\infty} \frac{x^k}{k!} (s_k - s_N).$$

Let $\lambda$ be a constant such that (2.17) holds. Write

$$\mu(n) = \log n - (\log n)^i, \nu(n) = \log n + (\log n)^i.$$

Since, by (1.2)

$$s_k - s_N = o(k)$$
uniformly for $k \geq N$, it follows from Theorem 137 (6) of Hardy [5] that

$$e^{-x} \sum_{k \leq x \leq n} \frac{x^k}{k!} (s_k - s_N) = o(1).$$

Also, since
uniformly in $k \leq N$, it follows from Theorem 137 (3), loc. cit., that

$$e^{-k} \sum_{k \in P(n)} \frac{x^k}{k!} (s_k - s_N) = o(1).$$

Thus

$$f(x) = s_N + e^{-x} \sum_{\nu \in P(n) \subset k \subset P(n)} \frac{x^k}{k!} (s_k - s_N) + o(1).$$

We also have

$$\sigma_n = \frac{1}{n!} \sum_{k=1}^{n} p_{n,k}(s_N + (s_k - s_N)).$$

But Agnew ([1], p. 106) has remarked that

$$\frac{1}{n!} \sum_{k=1}^{n} p_{n,k} = \frac{1}{n!} p_s(1) = 1.$$

Hence

$$\sigma_n = s_N + \frac{1}{n!} \sum_{k=1}^{n} p_{n,k}(s_k - s_N).$$

Let $b$ be a constant such that $b \geq 1$ and such that, with the notation of (2.5),

$$\phi(b) < -2.$$

It is possible to choose such a constant, since

$$\phi(\theta) \longrightarrow -\infty \text{ as } \theta \longrightarrow \infty.$$

It follows from (2.30) and (2.31) that

$$\sigma_n - f(\log n) = \left( \sum_{1 \leq k \leq \log n} + \sum_{\nu \in P(n) \leq k \leq \log n} + \sum_{k \geq \log n} \right) \frac{p_{n,k}}{n!} (s_k - s_N)$$

$$+ \sum_{\nu \in P(n) \subset k \subset P(n)} \left( \frac{p_{n,k}}{n!} - e^{-x} \frac{x^k}{k!} \right) (s_k - s_N) + o(1)$$

$$= \sum_{1} + \sum_{2} + \sum_{3} + \sum_{4} + o(1),$$

say, where $x = \log n$.

It follows from Lemma 2.1 that, for all terms occurring in the sum $\Sigma_i$, the value of $p_{n,k}/n!$ is less than the value it takes for the last term, and by Lemma 2.3 this is

$$O\left\{ \frac{1}{\sqrt{\log n}} \exp \left[ -\frac{1}{2} (\log n)^{2i-1} \right] \right\}.$$
Since the number of terms in the sum is $O(\log n)$, it follows with the aid of (2.28) that

$$\sum_i = o(1).$$

We can deal with $\sum_s$ in a similar way. Again for all terms occurring in the sum $\sum_s$, the value of $p_{sk}/n!$ is less than the value it takes for the first term, and by Lemma 2.2 this is

$$O\left(\frac{1}{\log n} n^{q(s)}\right).$$

We have, for each individual term

$$s_k - s_N = o(n)$$

and the number of terms in the sum does not exceed $n$; hence it follows with the aid of (2.32) that

$$\sum_s = o\left\{\frac{n^{q(s)+2}}{\sqrt{\log n}}\right\} = o(1).$$

It follows from Lemma 2.3 and from Theorem 137 (5) of Hardy [5] that in the range of summation of $\sum_s$ we have, with $x = \log n$, $h = k - \log n$

$$\frac{p_{sk}}{n!} - e^{-x} \frac{x^k}{k!} = \frac{1}{\sqrt{\log n}} \left[ \exp\left(\frac{-h^2}{2 \log n}\right) \right] \left[ O\left(\frac{|h| + 1}{\log n}\right) \right] + O\left(\frac{|h|^3}{\log^2 n}\right).$$

Further, in this range it follows from (1.2) that

$$s_k - s_N = o(h).$$

Further,

$$|h| + 1 = o(|h|)$$

except for the term $k = N$, since $|h| \geq \frac{1}{2}$; and, for this term $s_k - s_N$ vanishes. Hence

$$\sum_i = o\left\{\frac{1}{\sqrt{\log n}} \sum_{p_{sk} < h \leq \varphi(n)} \chi(h)\right\}$$

where

$$\chi(h) = \chi(h; n) = |h| \left(\frac{|h|}{\log n} + \frac{|h|^3}{\log^2 n}\right) \exp\left(-\frac{h^2}{2 \log n}\right).$$

It is easily verified that, for $h > 0$, $\chi(h)$ is increasing for $h < h_0 = h_0(n)$ (say) and decreasing for $h > h_0$. Thus for any integer $k$ with
\[ h = k - \log n \leq h_0 - 1 \]

we have

\[ \chi(h) < \int_{h}^{h+1} \chi(t) dt, \]

(2.34)

and similarly for \( h \geq h_0 + 1 \).

\[ \chi(h) < \int_{h-1}^{h} \chi(t) dt. \]

(2.35)

There are at most two terms for which neither of the inequalities (2.34), (2.35) are valid; and these are \( O(1) \) (uniformly in \( n \)) since \( \chi(h; n) \) is bounded. We can deal with negative values of \( h \) in a similar way. It thus follows from (2.27) that expression in curly brackets in (2.33) does not exceed

\[ \int_{-(\log n)^2}^{(\log n)^2} \chi(h) dh + O\left( \frac{1}{\sqrt{\log n}} \right). \]

Using this in (2.33) it follows that

\[ \sum_{n} = O\left\{ \frac{1}{\sqrt{\log n}} \int_{-(\log n)^2}^{(\log n)^2} \left( \frac{h^2}{\log n} + \frac{h^4}{\log n} \right) \exp \left( -\frac{h^2}{2 \log n} \right) du \right\} \]

\[ = O\left\{ \int_{-\infty}^{\infty} (u^2 + u^4) \exp \left( -\frac{u^2}{2} \right) du \right\}. \]

This is enough to establish (2.1).

3. Theorem 3.1. Suppose that

\[ a_k = O\left( \frac{1}{k^3} \right). \]

(3.1)

Let \( m \) be an integer valued function of \( n \) such that

\[ \lim sup |(m - \log n)/\sqrt{\log n}| \leq c, \]

(3.2)

where \( c \) is a constant. In other words

\[ m = \log n + c\sqrt{\log n} + o(\sqrt{\log n}). \]

Then

\[ \lim sup \left| \sigma_n - s_m \right| \leq \phi(c) \lim sup \left| \frac{1}{k^3} a_k \right|, \]

(3.4)

where \( \phi(c) \) is a Tauberian constant defined by

\[ \phi(c) = \sqrt{\frac{2}{\pi}} \left\{ \exp (-c^2/2) + c \int_{0}^{c} \exp (-u^2/2) du \right\}. \]

(3.5)

The result is the best possible in the sense that equality can occur.
The least possible value of \( \phi(c) \) occurs when \( c = 0 \).

Theorem 3.1 follows at once from Agnew's result of ([2] §'s 2, 3) with the aid of Theorem 2.1. It also could be deduced from Theorem 1 of Meir² [9], since Lemma 2.3 satisfies Meir's conditions when Meir's \( q \) equals \( \log n \).

Theorem 3.1 implies analogous results to Theorem 1.4, 1.5 of Agnew [2] but for the Lototsky transform instead. The analogue of Agnew's result of ([2] § 4) for the Lototsky transform can be deduced from Agnew's result of § 4 with the aid of Theorem 2.1. The only change in our results is that we have \( \log n \) instead of Agnew's \( t \).

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² Meir states in his Lemma B that the other conditions imply his Equation (3.4). This is obviously untrue, but if we assume his Equation (3.4) as an additional hypothesis, then Meir's theorems become correct.
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