MAXIMAL NONNORMAL CHAINS IN FINITE GROUPS

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In a finite group $G$, knowledge of the distribution of the subnormal subgroups of $G$ can be used, to some extent, to describe the structure of $G$. Here we show that if $G$ is a finite nonnilpotent, solvable group such that every upper chain of length $n$ in $G$ contains a proper subnormal entry then:

1. the nilpotent length of $G$ is less than or equal to $n$.
2. $|G|$ has at most $n$ distinct prime divisors, furthermore if $|G|$ has $n$ distinct prime divisors, then $G$ has abelian Sylow subgroups.
3. if $|G|$ has at least $(n - 1)$ distinct prime divisors, then $G$ is a Sylow Tower Group, for some ordering of the primes.
4. $r(G) \leq n$, where $r(G)$ denotes the minimal number of generators for $G$.

Before proving these results it is necessary to have a few lemmas concerning upper chains and subnormal subgroups. All groups are assumed to be finite.

An upper chain of length $r$ in $G$ is a sequence of subgroups, $G = G_0 \supset G_1 \supset \cdots \supset G_r$ where for each $i$, $G_i$ is maximal in $G_{i-1}$. Janko [4] has described the finite groups in which every upper chain of length four terminates in a normal subgroup. We define the function $h(G)$ as follows:

**Definition 1.** $h(G) = n$ if every upper chain in $G$ of length $n$ contains a proper ($\neq G$) subnormal entry and there exists at least one upper chain of length $(n - 1)$ which contains no proper subnormal entry.

Note that since a subnormal maximal subgroup is normal, $h(G) = 1$ if and only if $G$ is nilpotent. From the definition it is clear that if $h(G) = n$ then there exists an upper chain of length $n$ such that only the terminal entry is subnormal in $G$. Such a chain is called an $h$-chain for $G$. The following two lemmas are simple modifications of Lemmas 2, 3 [2].

**Lemma 1.** If $H$ is a nonnormal maximal subgroup of $G$, then $h(H) \leq h(G) - 1$.

**Lemma 2.** If $N$ is a normal subgroup of $G$, then $h(G/N) \leq h(G)$.

**Lemma 3.** If $G = H \times K$, where $h(H) \geq 2$, then $h(G) \geq h(H) + m$, where $m$ is the number of primes dividing $|K|$.
where \( m \) is the length of the longest chain in \( K \).

Proof. Let \( H = H_0 \supset H_1 \supset \cdots \supset H_r \) be an \( h \)-chain for \( H \) and \( K = K_0 \supset K_1 \supset \cdots \supset K_m = \langle 1 \rangle \) be the longest chain in \( K \). Then in \( H \times K \) the upper chain:

\[
H_0 \times K_0 \supset H_1 \times K_0 \supset H_1 \times K_1 \supset H_1 \times K_2 \supset \cdots \supset H_1 \times K_m = H_1 \supset H_2 \cdots \supset H_r ,
\]

has \((r + m)\) entries. If one of these entries is subnormal in \( G \), then its projection on \( H \) is subnormal in \( H \). However these projections are simply \( H_1, H_2, \cdots, H_r \), and of these, only \( H_r \) is subnormal in \( H \). Thus \( h(H \times K) \geq r + m \).

For reference it is convenient to note here the notion of a Saturated Formation as defined by Gaschutz [3].

**Definition 2.** A formation \( \mathcal{F} \) is a collection of finite solvable groups satisfying:

1. \( \langle 1 \rangle \in \mathcal{F} \).
2. If \( G \in \mathcal{F} \), and \( N \triangleleft G \), then \( G/N \in \mathcal{F} \).
3. If \( G/N_i \in \mathcal{F} \), \( i = 1, 2 \), then \( G/(N_1 \cap N_2) \in \mathcal{F} \).

A formation \( \mathcal{F} \) is called saturated if given a group \( G \) which does not belong to \( \mathcal{F} \), if \( M \) is a minimal normal subgroup of \( G \), such that \( G/M \in \mathcal{F} \), then \( M \) has a complement in \( G \), and all such complements are conjugate. Gaschütz showed later that conjugacy follows from existence and furthermore saturation can be characterized as follows:

A formation \( \mathcal{F} \) is saturated if whenever \( G/\phi(G) \) belongs to \( \mathcal{F} \) then \( G \) also belongs to \( \mathcal{F} \), where \( \phi(G) \) denotes the Frattini subgroup of \( G \). The collection of all finite solvable groups constitutes a formation, as does the collection of all finite nilpotent groups. This can be extended in a natural way to a theorem on all groups having a given bound on nilpotent length. By the nilpotent length (denoted by \( l(G) \)) of a solvable group we mean the length of the shortest normal chain with nilpotent factors. Example 4.5 [3] shows that the set, \( \mathcal{F}_n \), of all solvable groups \( G \) such that the nilpotent length of \( G \) is less than or equal to \( n \) is a saturated formation for each \( n \).

Theorem 1 shows the relation between \( h(G) \) and \( l(G) \).

**Theorem 1.** If \( G \) is a solvable group then \( l(G) \leq h(G) \).
Proof. The proof is by induction on $h(G)$, the theorem being trivially true if $h(G) = 1$. So suppose the theorem is true for all groups $K$ such that $h(K) \leq (n - 1)$ and is false for some group $K$ where $h(K) = n$. Among such groups let $G$ be one of minimal order. We show that such a group $G$ cannot exist. Let $M$ be a minimal normal subgroup of $G$. By Lemma 2, $h(G/M) \leq h(G) = n$ so that by the minimality of $G$, $l(G/M) \leq n$. If $N$ is another minimal normal subgroup of $G$, then by the same argument $l(G/N) \leq n$. By the saturated formation property $l(G/(M \cap N)) \leq n$. Since $M \cap N = \langle 1 \rangle$, this is impossible, so $M$ is the unique minimal normal subgroup of $G$. By the saturated formation property and minimality of $G$, $M$ has a complement $L$ in $G$. $G = ML$, $M \cap L = \langle 1 \rangle$. Since $M$ is the unique minimal normal subgroup of $G$, $L$ is a nonnormal, maximal subgroup. By Lemma 1 $h(L) \leq (n - 1)$. Hence by the induction hypothesis, $l(L) \leq (n - 1)$. Since $L \cong G/M$ and $M$ is abelian $l(G) \leq n$. This is a contradiction, therefore $G$ does not exist.

By looking at the holomorph of a group of prime order $p$ where $p = 2^nk + 1$ we see that no converse to Theorem 1 is possible, i.e., it is possible to have $l(G) = 2$ and $h(G)$ arbitrarily large.

For notation purposes let $\pi(G : K)$ denote the number of distinct prime divisors of $[G : K]$, with $\pi(G : \langle 1 \rangle)$ denoted simply by $\pi(G)$. Then there is a relationship between $h(G)$ and $\pi(G)$.

**Theorem 2.** If $G$ is a solvable group such that $h(G) < \pi(G)$ then $h(G) = 1$, i.e., $G$ is nilpotent.

**Proof.** Suppose the theorem is false and let $G$ be a counter-example. Let $P$ be a nonnormal Sylow subgroup of $G$. Consider an upper chain from $G$ through $N_o(P)$ to $P$. Since $G$ is solvable this chain is at least $(\pi(G) - 1)$ entries long. Thus by hypothesis this chain must contain a subnormal entry. However $N_o(P)$ is not contained in a proper subnormal subgroup, and if $N_o(P)$ contains a subnormal subgroup containing $P$, $P$ is subnormal. But a subnormal Sylow subgroup is normal. Thus we have a contradiction so $G$ cannot exist.

$S_3$, the symmetric group on three symbols, has: $h(S_3) = \pi(S_3) = 2$, showing that the arithmetic condition of Theorem 2 cannot be relaxed. However this does suggest the question of what structure follows from the hypothesis that $h(G) - \pi(G)$ is small. $G$ is called a *Sylow Tower Group* (STG) if $G$ has a normal Sylow subgroup, and every homomorphic image of $G$ has a normal Sylow subgroup.
THEOREM 3. If $G$ is solvable and $h(G) - \pi(G) \leq 1$, then $G$ is a Sylow Tower Group for some ordering of the prime divisors of $G$.

Proof. The proof is by induction on $h(G)$, the theorem being trivially true if $h(G) = 1$. Suppose the theorem is true for all groups $K$ for which $h(K) < n$, and is false for some group $K$ for which $h(K) = n$. Among such groups let $G$ be one of minimal order. We will show that $G$ cannot exist thereby proving the theorem. $G$ must satisfy the following:

(1) Every nonnormal maximal subgroup of $G$ is STG.

Let $H$ be a nonnormal maximal subgroup of $G$. $\pi(G : H) = 1$ so $\pi(H) \geq (n - 2)$. By Lemma 1, $h(H) \leq (n - 1)$. Thus by the induction hypothesis $H$ is STG.

(2) $G$ does not possess a normal Sylow subgroup.

Suppose $P$ is a normal Sylow subgroup of $G$. Let $K$ be a subgroup maximal with respect to the properties: i) $P \triangleleft K \triangleleft G$, and $G/K$ does not possess a normal Sylow subgroup since $K$ is maximal with respect to the property of being STG. $K$ is a normal Hall subgroup so $K$ has a complement $L$. $L \cong G/K$ so $L$ is not STG. $L$ is Hall so $N(L)$ is abnormal, so if $N(L) \not\cong G$, $N(L)$ is contained in an abnormal maximal subgroup whence by (1) is STG. This contradicts the fact that $L$ is not STG, so $N(L) = G$, and $G = H \times L$. Suppose $\pi(K) = m$, then $\pi(L) = \pi(G) - m$ so $h(L) \geq \pi(G) - m + 2$ by induction. Hence by Lemma 3, $h(G) \geq (\pi(G) - m + 2) + m = \pi(G) + 2$ which is a contradiction, so $P$ does not exist.

(3) $G$ possesses a unique minimal normal subgroup $M$; furthermore $G/M$ is supersolvable.

Let $M$ be a minimal normal subgroup of $G$. By (2), $M$ is not a Sylow subgroup. Thus $\pi(G/M) = \pi(G)$. $h(G/M) \leq h(G)$ so by the minimality of the order of $G$, $G/M$ is STG. Now the groups having a Sylow tower for a given ordering of the primes constitute a saturated formation [1]. Thus $M$ has a complement $L$ in $G$, and $L$ is STG. Let $L = L_1 \triangleright L_2 \triangleright \cdots \triangleright L_{n-1} \triangleright L_n \triangleright \cdots \triangleright \langle 1 \rangle$ be a Sylow tower for $L$. We refine this chain and adjoin $G$ to obtain an upper chain. If for any $i < n$, $L_{i-1}/L_i$ is not simple, $L_n$ is subnormal in $G$. However this will give rise to a normal Sylow subgroup in $G$, contradicting (2). Hence each $L_{i-1}/L_i$ is of prime order and $L_n$ is cyclic. Hence $L$ is supersolvable. We have shown that the factor group to a minimal normal subgroup is supersolvable. Therefore if $G$ has two distinct minimal normal subgroups $N_1$ and $N_2$, then $G/N_1$ is supersolvable $i = 1, 2$, so that $G/(N_1 \cap N_2)$ is supersolvable. Since $N_1 \cap N_2 = \langle 1 \rangle$ this implies that $G$ is supersolvable. However supersolvable groups are STG, so $M$ is unique.
Using the same notation as in (3), since \( L \) does not contain a nontrivial normal subgroup, \( L \) does not contain a nontrivial subnormal subgroup thus from the chain obtained above we see that \(| L |\) is square free.

Since \( L \) is supersolvable we may assume that the Sylow subgroup for the largest prime is normal in \( L \). Let \(| M | = p^\alpha, \ p \) prime. Suppose \( Q \) is a Sylow \( q \)-subgroup of \( G \) where \( q \) is the largest prime divisor of \(| G |\). We may assume \( p \neq q, Q < L, \) in fact \( N(Q) = L \).

\((4)\) \(| G | = 24, h(G) = 3.\)

Let \( P \) be a Sylow \( p \)-subgroup of \( G \). Then since \(| L |\) is square free, \(| P | = | M | \cdot p.\)

We may assume that \( P \) contains a Sylow \( p \)-subgroup \( T \) of \( L \). Then since \( T \) is not subnormal, \( P \) contains a maximal (in \( P \)) non-subnormal (in \( G \)) subgroup \( J. \) \( P = MJ, [P : M \cap J] = p^i. \) Now \( J \) is \((n - 1)\)-th maximal and not subnormal, and \( h(G) = n, \) thus each maximal subgroup of \( J \) is subnormal in \( G \). Hence \( J \) has just one maximal subgroup, and so \( J \) is cyclic. However \( M \) is elementary abelian, therefore \(| M \cap J | = 1 \) or \(| M \cap J | = p. \) Thus \(| M | = p \) or \( p^i. \) However \(| M | = [G : L] \equiv 1 \pmod{q}, \) by the Sylow theorems. Now \( p < q \) so \(| M | = p^i. \) Since \( q | (p^i - 1), q = p + 1, \) so that \( q = 3, p = 2, \) and \(| G | = 24, h(G) = 3. \)

\((5)\) The final contradiction.

Note that \( G \) is not \( S_4 \) since \( h(S_4) = 4. \) Now in \( G \) the subgroups of order 2 are subnormal. Thus the normalizer of the Sylow 3-subgroup is cyclic. By Burnside's theorem the 3-Sylow subgroup has a normal complement contrary to (2). Thus \( G \) does not exist.

Note that \( h(S_i) = 4, \pi(S_i) = 2 \) and \( S_i \) is not STG.

In the special case where \( h(G) = \pi(G), \) even more can be said.

**Theorem 4.** If \( G \) is solvable and \( h(G) = \pi(G) \geq 2, \) then the Sylow subgroups of \( G \) are cyclic or elementary abelian. Furthermore if there exist at least two nonisomorphic nonnormal Sylow subgroups of \( G, \) then all nonnormal Sylow subgroups of \( G \) are of prime order.

**Proof.** Let \( \pi(G) = h(G) = n. \) Let \( P \) be a nonnormal Sylow subgroup of \( G. \) As in Theorem 2, \( \pi(G : P) = (n - 1) \) so that \( P \) is at least \((n - 1)\)-th maximal in \( G. \)

Considering a chain through \( N(P) \) to \( P, \) as in the proof of Theorem 2 we see that this chain can have at most \((n - 1)\) entries, hence exactly \((n - 1)\) entries. Therefore \( P \) is cyclic, since every maximal subgroup of \( P \) is subnormal in \( G, \) and \( P \) is not. In this chain we have \((n - 1)\) distinct primes and \((n - 1)\) entries. Therefore each entry
is a Sylow complement in its predecessor. However this implies that
the Sylow subgroup is elementary abelian. If there were two non-
normal Sylow subgroups, then by this same argument \( P \) is elementary
abelian. However \( P \) is cyclic so that \( P \) is of prime order.

Note that under the hypothesis of Theorem 4, if we let \( K \) denote
the product of all the normal Sylow subgroups in \( G \), then \( K \) is abelian
and \( G/K \) has cyclic Sylow subgroups, so that \( l(G) \leq 3 \). Also we should
note that an extension of the Quaternion group of order 8 by an
automorphism which permutes the subgroups of order 4 will yield a
non-A-group \( G \) having \( h(G) = 3 \) and \( \pi(G) = 2 \).

To see how these theorems restrict the structure of a solvable
group in a particular case, consider the groups \( G \) having \( h(G) = 2 \).

THEOREM 5. Suppose \( h(G) = 2 \). Then \( G = PQ; P \) and \( Q \) are
Sylow subgroups of \( G \); \( P \) is a minimal normal subgroup; \( Q \) is cyclic;
\( Q_1 \), the maximal subgroup of \( Q \), is normal in \( G \), in fact, \( Q_1 = \phi(G) = Z(G) \).

Note that a theorem due to Rose [5] shows that \( h(G) = 2 \) implies
solvability for \( G \). More generally, we can effectively duplicate the
proofs of the theorems in [2] to prove:

THEOREM 6. If \( G \) is a finite group, and \( h(G) \leq 3 \), then \( G \) is
solvable. Moreover if \( h(G) \leq 4 \) and \((|G|, 3) = 1 \), then \( G \) is solvable.

Note that \( A_6 \), the simple group of order sixty, has \( h(A_6) = 4 \).
The groups described in Theorem 5 have the property that they
can be generated by two elements. This can be extended to a more
general theorem.

Let \( r(G) \) denote the minimal number of generators for \( G \).

THEOREM 7. If \( h(G) \geq 2 \), then \( r(G) \leq h(G) \).

Proof. The condition \( h(G) \geq 2 \) is certainly necessary since we
can find abelian groups \( K \) with \( r(K) \) large. To prove Theorem 7 we
only need to note that the next to last entry in an \( h \)-chain for \( G \) is
\((h(G) - 1)\)-th maximal in \( G \) and is cyclic.

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