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## **ALGEBRAIC PROPERTIES OF CERTAIN RINGS OF CONTINUOUS FUNCTIONS**

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**Let  $X$  and  $Y$  be any subsets of  $E^n$ , and  $(X', d_1)$  and  $(Y', d_2)$  be any metric spaces. Let  $C^m(X)$ ,  $0 \leq m \leq \infty$ , denote the ring of  $m$ -differentiable functions on  $X$ , and  $L_c(X')$  be the ring of the functions which are Lipschitzian on each compact subset of  $X'$ , and  $L(X')$  be the ring of the bounded Lipschitzian functions on  $X'$ . The relations between algebraic properties of  $C^m(X)$ , (resp.  $L_c(X')$  or  $L(X')$ ) and the topological properties of  $X$  (resp.  $X'$ ) are studied. It is proved that if  $X$  and  $Y$ , (resp.  $(X', d_1)$  and  $(Y', d_2)$ ) are  $m$ -realcompact, (resp.  $L_c$ -real-compact or compact) then  $C^m(X) \cong C^m(Y)$  (resp.  $L_c(X') \cong L_c(Y')$  or  $L(X') \cong L(Y')$  if and only if  $X$  and  $Y$  are  $C^m$ -diffeomorphic (resp.  $(X', d_1)$  and  $(Y', d_2)$  are  $L_c$  or  $L$ -homeomorphic).**

During the last twenty years, the relations between the algebraic properties of  $C^m(X)$  and  $C^m(Y)$  and the topological properties of  $X$  and  $Y$  have been investigated by Hewitt [4], Myers [9], Pursell [11], Nakai [10], and Gillman and Jerison [3], where  $m$  is a positive integer, zero or infinite. In 1963, Sherbert [12] studied the ring  $L(X)$ . Recently, Magill, [6] has obtained the algebraic condition relating  $C(X)$  and  $C(Y)$  (i. e.,  $m = 0$ ) which are both necessary and sufficient for embedding  $Y$  in  $X$ , where  $X$  and  $Y$  are two realcompact spaces.

This work is to utilize the method of Gillman and Jerison [3] for studying the algebraic properties of  $C^m(X)$  and  $L_c(X_1)$  (§§ 2-5), and how they are related with topological properties of  $X$  and  $X_1$  respectively. In view of [8, Cor. 1.32], we will restrict  $X$  in  $C^m(X)$  to a subset of  $E^n$ . The results of Magill are also true in  $C^m(X)$  and  $L_c(X)$  with some modification. In the last section, § 6, we observe some other cases.

2. Rings and ideals. Let  $X$  be an arbitrary subset of  $E^n$ , an  $n$ -dimensional euclidean space, and  $C^m(X)$  be the set of all real-valued functions of class  $C^m$  in the sense of Whitney [14, § 3], where  $m$  will always refer to an arbitrary integer such that  $0 \leq m \leq \infty$ . By [15, Th. 4], we know that  $C^m(X)$  forms a ring with the identity  $u$ , the constant function of value 1, and zero element  $\theta$ , the constant function of value 0. Let  $C^{m*}(X) = \{f \in C^m(X) : f \text{ is bounded}\}$ . It is clear that  $C^{m*}(X)$  is a subring of  $C^m(X)$  with  $u$  and  $\theta$ . Let  $X$  be a metric space, and  $L_c(X)$  be the set of all real-valued functions satisfying Lipschitz condition on each compact subset of  $X$  [2, p. 354]. We can easily show that  $L_c(X)$  is a ring with  $u$  and  $\theta$ . Let  $L(X) =$

$\{f \in L_c(X) : f \text{ is bounded and Lipschitzian on entire } X\}$ ,  $L_c^*(X) = \{f \in L_c(X) : f \text{ is bounded}\}$ . Then, both  $L(X)$  and  $L_c^*(X)$  are the sub-rings of  $L_c(X)$  with  $u$  and  $\theta$ .

Since the properties of  $C^m(X)$  (resp.  $C^{m*}(X)$ ) and those of  $L_c(X)$  (resp.  $L_c^*(X)$  and  $L(X)$ ) are almost all the same, we will use  $\mathfrak{U}$  and  $\mathfrak{U}'$  to denote  $C^m(X)$  (resp.  $L_c(X)$ ) and  $C^m(Y)$  (resp.  $L_c(Y)$ ), and  $\mathfrak{B}$  and  $\mathfrak{B}'$  to denote  $C^{m*}(X)$  (resp.  $L_c^*(X)$ , and  $L(X)$ ) and  $C^{m*}(Y)$  (resp.  $L_c^*(Y)$  and  $L(Y)$ ) respectively, where  $X$ , and  $Y$  are appropriately the subsets of  $E^n$  or metric space. Also “ $a$ -” and “ $b$ -” will mean  $m$ - (or  $C^m$ -) (resp.  $L_c$ ) and  $C^{m*}$ - (resp.  $L_c^*$  and  $L$ ) respectively according as  $\mathfrak{U}$  is  $C^m(X)$  (resp.  $L_c(X)$ ) and  $\mathfrak{B}$  is  $C^{m*}(X)$  (resp.  $L_c^*(X)$ , and  $L(X)$ ).

The unit element of an  $f \in \mathfrak{U}$  or  $\mathfrak{B}$  is defined as usual. For  $f \in \mathfrak{U}$ ,  $Z(f) = \{x \in X : f(x) = 0\}$  is said to be the zero-set of  $f$ .  $Z(\mathfrak{U}) = \{Z(f) : f \in \mathfrak{U}\}$ . It is then clear that  $f \in \mathfrak{U}$  is a unit if and only if  $Z(f) = \emptyset$ . (For  $C^m(X)$  see [15, Th. 4].) Likewise, if  $f \in \mathfrak{B}$  is a unit, then  $Z(f) = \emptyset$ . But the converse need not hold, for the multiplicative inverse  $1/f$  of  $f$  in  $\mathfrak{U}$  may not be a bounded function. For example: let  $X = E^1$ , and  $f(x) = e^{-x^2} \in C^{m*}(E^1)$  and  $Z(f) = \emptyset$ . But  $1/f = e^{x^2} \notin C^{m*}(E^1)$ .

A  $z$ -filter of  $Z(\mathfrak{U})$  is the same as in [3,2.2]. It is obvious that  $Z[I] = \{Z(f) : f \in I\}$  is a  $z$ -filter on  $X$  if  $I$  is a proper ideal in  $\mathfrak{U}$ , and  $Z^{-1}[\mathcal{F}] = \{f \in \mathfrak{U} : Z(f) \in \mathcal{F}\}$  is a proper ideal if  $\mathcal{F}$  is a  $z$ -filter on  $X$ . Note that it may be false that a proper ideal  $I \subset \mathfrak{B}$  implies that  $Z[I]$  is a  $z$ -filter. For example: let us consider  $\mathfrak{B} = C^{m*}(E^1)$  and let  $f(x) = 1/(1 + x^2)$ , and  $I = (f)$  be the ideal generated by  $f$  in  $\mathfrak{B}$ . Then it is clear that  $\emptyset \in Z[I]$ .

Hereafter, we will always use “ideal” to mean the proper ideal, unless the contrary is mentioned.

Accordingly, every  $z$ -filter is of the form  $Z[I]$ , for some ideal  $I$  in  $\mathfrak{U}$ . That  $Z^{-1}[Z[I]] \supset I$  is also clear. The inclusion may be proper.

For instance, consider  $\mathfrak{U} = C^m(E^1)$ . (a) For any positive integer  $m$ , let  $i(x) = x$  for all  $x \in E^1$ , and  $I = (i)$ . Then

$$Z^{-1}[Z[I]] = M_0 = \{f \in C^m(E) : f(0) = 0\}.$$

However,  $i^{(3m+1)/3} \in M_0 - I$ . (b) In case  $m = \infty$ , let  $f_1(x) = e^{-1/x^2}$  for  $x \in E^1$  and  $I_1 = (f_1)$ . Then  $M_0 = Z^{-1}[Z[I]]$  contains an element  $i \in I_1$ . Note that  $M_0$  is a maximal fixed ideal. Now, as for  $L_c(X)$ , we may consider  $(X, d)$  to be a bounded metric space, and  $f_0(x) = (f_p(x))^2 = (d(p, x))^2$ . Then  $f_0 \in L_c(X)$ . Let  $I_0 = (f_0)$ . Then

$$Z^{-1}[Z[I_0]] = \{f \in L_c(X) : f(p) = 0\} = M_p$$

is clear. However,  $f_p(x) = d(p, x) \in M_p - I_0$ .

A  $z$ -ultrafilter on  $X$  is a maximal  $z$ -filter [3,2.5]. We know that every subfamily of  $Z(\mathfrak{A})$  with the finite intersection property, by Zorn's Lemma, is contained in some  $z$ -ultrafilter on  $X$ .

The proofs of following propositions are obvious.

**PROPOSITION 2.1.** If  $M$  is a maximal ideal in  $\mathfrak{A}$ , then  $Z[M]$  is a  $z$ -ultrafilter on  $X$ .

**PROPOSITION 2.2.** If  $\mathscr{A}$  is a  $z$ -ultrafilter on  $X$ , then  $Z^{-1}[\mathscr{A}]$  is a maximal ideal in  $\mathfrak{A}$ .

It follows from Propositions (2.1) and (2.2) that the mapping  $Z$  is one-one from the set of all maximal ideals in  $\mathfrak{A}$  onto the set of all  $z$ -ultrafilters on  $X$ .

**PROPOSITION 2.3.** Let  $M$  be a maximal ideal in  $\mathfrak{A}$ . If  $Z(f)$  meets every member of  $Z[M]$ , then  $f \in M$ .

**PROPOSITION 2.4.** Let  $\mathscr{A}$  be a  $z$ -ultrafilter on  $X$ . If a zero-set  $Z$  meets every member of  $\mathscr{A}$ , then  $Z \in \mathscr{A}$ .

An ideal  $I$  in  $\mathfrak{A}$  is  $z$ -ideal if  $Z(f) \in Z[I]$  implies  $f \in I$ . That is,  $I = Z^{-1}[Z[I]]$ , [3,2.7]. It is obvious that every maximal ideal is a  $z$ -ideal. A prime ideal is defined in the usual sense. The following theorem is only true for  $L_c(X)$ ,  $L_c^*(X)$  or  $L(X)$ . For we can show that these are lattice-ordered rings; while  $C^m(X)$  and  $C^{m*}(X)$  are not.

**THEOREM 2.5.** For any  $z$ -ideal  $I$  in  $L_c(X)$  ( $L_c^*(X)$  or  $L(X)$ ) the following are equivalent:

- (1)  $I$  is prime.
- (2)  $I$  contains a prime ideal.
- (3) For all  $g, h \in L_c(X)$  ( $L_c^*(X)$  or  $L(X)$ ),  $g \cdot h = \theta$ , then  $g \in I$  or  $h \in I$ .
- (4) For every  $f \in L_c(X)$  ( $L_c^*(X)$  or  $L(X)$ ), there is a zero-set in  $Z[I]$  on which  $f$  does not change sign.

Proof is similar to [3,2.9].

**3. Zero-set,  $a$ -completely regular and  $a$ -normal spaces.** We know from the proof of Lemma 25 [16, p. 669] that each closed subset  $F$  of  $E^n$ , there is an  $f \in C^m(X)$  such that  $Z(f) = F$ .

**PROPOSITION 3.1.** For each closed subset  $A$  of  $(X, d)$ , there is  $f \in L_c(X)$  (in fact  $f \in L(X)$ ) such that  $Z(f) = A$ .

*Proof.* Let  $g(x) = d(A, x)$  and  $f = g \wedge u^1$ . Then  $f \in L(X)$  and  $Z(f) = A$ .

DEFINITION 3.2. Let  $X$  be a topological space.  $X$  is said to be  $a$ -completely regular if and only if for each closed subset  $F$  of  $X$  and  $x \notin F$ , there is an  $f \in \mathfrak{A}$  such that  $f(x) = 1$ , and  $f[F] = \{0\}$ .

THEOREM 3.3. A topological space is  $a$ -completely regular if and only if the family  $Z(\mathfrak{A}) = \{Z(f) : f \in \mathfrak{A}\}$  is a base for the closed subsets of  $X$ .

Proof is similar to [3, 3.2].

DEFINITION 3.4. A topological space is said to be  $a$ -normal if for any disjoint closed subsets  $F_1$  and  $F_2$ , there is an  $f \in \mathfrak{A}$  such that  $f[F_1] = \{0\}$  and  $f[F_2] = \{1\}$ .

PROPOSITION 3.5. Every subset  $X$  of  $E^n$  is  $m$ -normal. Hence is  $m$ -completely regular.

*Proof.* Let  $F_1$  and  $F_2$  be any two disjoint closed subsets of  $X$ . We know that there are closed subsets  $F'_1$  and  $F'_2$  of  $E^n$  such that  $F_i = F'_i \cap X$ ,  $i = 1, 2$ . We know that there are  $f_i \in C^m(E^n)$  with  $Z(f_i) = F'_i$ . Let  $g_i = f_i|X$ , and  $f = g_1^2/(g_1^2 + g_2^2)$ . Then  $f \in C^m(X)$  and  $Z(f) = Z(g_1) = F_1$ ,  $f[F_2] = \{1\}$ . The last part is obvious.

PROPOSITION 3.5'. Every metric space  $(X, d)$  is  $L$ -completely regular; and every compact metric space  $(X, d)$  is  $L$ -normal.

*Proof.* Let  $F$  be a closed subset of  $X$  and  $p \in X - F$ . Then  $d(F, p) \neq 0$ . Let  $f$  be defined as follows:  $f[F] = \{1\}$ , and  $f(p) = 0$ . Then  $f$  is bounded by 1 satisfies a Lipschitz condition with constant  $K = (d(F, p))^{-1}$  on  $F \cup \{p\}$ . We know that there exists  $f_0 \in L(X)$  such that  $f_0|F \cup \{p\} = f$ , (by [7, p. 97]). Hence the first assertion follows. The proof of the second part is the same.

Note that the compactness in (3.5)' cannot be omitted. For instance: let  $X = E^2$ ,

$$F = \{(x, y) \in E^2 : xy = 1\} \quad \text{and} \quad F' = \{(x, y) \in E^2 : xy = -1\}.$$

Then  $F$  and  $F'$  are two disjoint closed sets in  $E^2$ . However, it is clear that there is no  $f \in L(E^2)$  such that  $f[F] = \{1\}$  and  $f[F'] = \{0\}$ .

<sup>1</sup>  $u$  stands for the constant function of value 1.

Having done these, we can show the characterization of fixed maximal ideal of  $\mathfrak{A}$  and  $\mathfrak{B}$  and how they are related to a compact space are the same as in [3, (4.6), (4.8), (4.9) (a), (4.10) Lemma, and (4.11)].

4. **Real ideals,  $\alpha$ -realcompact space.** In 1948, E. Hewitt defined real maximal ideals and realcompact space ( $Q$ -spaces) (see [4, § 7] and [3, Ch. 5]). He also contributed many interesting properties about real maximal ideals and realcompact spaces. Unfortunately, those properties can only be carried to the rings  $L_c(X)$ ,  $L_c^*(X)$  and  $L(X)$ , but not to  $C^m(X)$ , since  $C^m(X)$  is not a lattice-ordered ring. (As  $f \in C^m(X)$  implies  $|f| \in C^m(X)$  is not always true.) Recently (1965), Rudolphe Bkouche has shown that every paracompact Hausdorff differentiable  $n$ -manifold is  $m$ -realcompact (see (4.2), and [1, Th. 2]). Here will show that every closed subset of  $E^n$  is  $m$ -realcompact.

We can show easily that every residue class field of  $\mathfrak{A}$  or  $\mathfrak{B}$  module a maximal ideal contains canonical copy of real field  $\mathbf{R}$ . We can also show, by using [3, (5.1) to (5.4)] or [13], that  $L_c/M$ ,  $L_c^*/M$ , and  $L(X)/M$  are totally ordered for each maximal ideal  $M$ . We will show that  $L_c^*(X)/M$  (resp.  $L(X)/M$ )  $\cong \mathbf{R}$  if  $M$  is maximal in  $L_c^*$  (resp.  $L(X)$ ). The real and hyper-real ideals are defined in [3, 5.9].

**LEMMA 4.1.** *Let  $M$  be a maximal ideal in  $L_c^*(X)$ , (resp.  $L(X)$ ), and  $L_c^*(X)$ , (resp.  $L(X)$ ) be normed by the sup norm  $\|\cdot\|_\infty$ . Then  $M$  is closed in  $L_c^*(X)$  (resp.  $L(X)$ ) under  $\|\cdot\|_\infty$ .*

*Proof.* In view of [3, 2 M1],  $\text{cl } M$  is either a proper ideal of  $L_c^*(X)$  or  $L_c^*(X)$  itself. Suppose  $\text{cl } M = L_c^*(X)$ . Then  $u \in \text{cl } M$ , and for any neighborhood of  $u$ ,  $N_\varepsilon(u)$ ,  $N_\varepsilon(u) \cap M \neq \emptyset$ . Take  $\varepsilon = 1/2$ . Then  $N_{1/2}(u) \cap M \neq \emptyset$ . That is, there is an  $f \in M$  such that

$$\|u - f\|_\infty < \frac{1}{2}.$$

This implies  $|f(x)| > 1/2$  for each  $x \in X$ . We can easily show that  $1/f \in L_c^*(X)$ . That is,  $M$  has a unit so that  $M = L_c^*(X)$ . This is a contradiction. Hence  $\text{cl } M$  is a proper ideal containing  $M$  so that  $M = \text{cl } M$ . The proof for  $M$  in  $L(X)$  is similar.

**PROPOSITION 4.2.** For each maximal ideal  $M$  in  $L_c^*(X)$ , (resp.  $L(X)$ ),  $L_c^*(X)/M$  (resp.  $L(X)/M$ )  $\cong \mathbf{R}$ .

*Proof.* It is enough to show that for any positive nonconstant residue class  $M(f)$ , simply denoted by  $f$ , there is a positive integer  $n$  such that  $f - 1/n$  is positive. (See [3, 5.6] and [13].) Suppose that

there does not exist such a positive integer. Then we would have  $f - 1/n$  is negative for all  $n \in N$ . That is,  $(f - 1/n) + |f - 1/n| \in M$  for all  $n$ . Consider now the sequence  $\{g_n = (f - 1/n) + |f - 1/n| : n \in N\}$  which has  $f + |f|$  as the limit under the norm  $\|\cdot\|_\infty$ . By (4.1),  $f + |f| \in \text{cl } M = M$ . This shows that  $-f \equiv |f| \pmod{M}$ . This is a contradiction.

**DEFINITION 4.3.** A topological space  $X$  is said to be  $\alpha$ -realcompact if every real maximal ideal in  $\mathfrak{A}$  is fixed.

It is clear that if  $X$  is compact, then  $X$  is  $\alpha$ -realcompact.

**LEMMA 4.4.** *An ideal in  $\mathfrak{A}$  is free if and only if for every compact subset  $A$  of  $X$  there exists an  $f \in I$  having no zero in  $A$ .*

*Proof.* Suppose  $I$  is free and  $A$  is any compact subset of  $X$ . If for each  $f \in I$ ,  $Z(f) \cap A \neq \emptyset$ , then  $\mathcal{F} = \{Z(f) \cap A : \text{for some } f \in I\}$  has the finite intersection property. Since  $A$  is compact,  $\bigcap \mathcal{F} \neq \emptyset$ . Hence  $\bigcap Z[I] \supseteq \bigcap \mathcal{F} \neq \emptyset$ , which is impossible.

The sufficiency is clear.

**PROPOSITION 4.5.** Let  $X$  be a closed subspace of  $E^n$ . Then  $X$  is  $m$ -realcompact as well as  $L_c$ -realcompact.

*Proof.* Suppose that  $M$  is a free maximal ideal and  $C^m(X)/M \cong \mathbf{R}$ . Let  $g(x) = 1/(\|x\|^2 + 1)$ . Then that  $g \in C^m(X)$  and  $g$  is a unit is clear. Hence  $g \notin M$ . i.e.,  $M(g) \neq 0$ . For any positive number  $r$  and a sufficiently small number  $\varepsilon > 0$ ,  $g < r - \varepsilon$  for all but a compact subset of  $E^n$ , say  $A_\varepsilon$ . Then  $B_\varepsilon = A_\varepsilon \cap X$  is compact in  $X$  as  $X$  is closed. Let  $A' = \text{cl}_X(X - B_\varepsilon)$ . Then there is an  $f \in C^m(X)$  such that  $Z(f) = A'$ . We will show that  $Z(f) \in Z[M]$ . We know that  $B_\varepsilon$  is compact in  $X$ . By (4.4), there is an  $f_1 \in M$  such that  $Z(f_1) \cap B_\varepsilon = \emptyset$ . Hence  $Z(f_1) \subseteq X - B_\varepsilon \subseteq Z(f)$  so that  $Z(f) \in Z(M)$ . Therefore,  $g \leq r - \varepsilon$  on the zero-set  $Z(f)$ , and  $r - g \geq \varepsilon$ . Let  $h_1 = (r - g)^{1/2}$  on  $Z(f)$ . Then  $h_1$  is  $C^m$  on  $Z(f)$  which is closed in  $E^n$ . By Whitney's Analytic Extension Theorem [14], we have a  $C^m$  extension  $h$ , i.e.,  $h|_{Z(f)} = h_1$ . Hence  $h^2 = r - g$  on  $Z(f)$ . Therefore,  $h^2 \equiv r - g \pmod{M}$ . In other words,  $M(h^2) = M(r - g) = M(r) - M(g) = r - M(g)$ . But, since  $C^m(X)/M$  is real  $M(h^2) \geq 0$ , we have  $M(g) \leq r$ . As  $r$  is any positive number,  $M(g)$  is infinitely small. This is a contradiction. The proof of the last part is similar.

We now will give an example to show that a nonparacompact space may not be an  $m$ -realcompact space. However, the existence of non- $L_c$ -realcompact spaces remains as an open question.

Let  $L$  be the long line as defined in [5]. Then,  $L$  is Hausdorff

space satisfying the first axiom of countability. Furthermore we have:

**PROPOSITION 4.6.** For each  $\alpha \in L$ ,  $\alpha \neq 1$ ,  $[1, \alpha]$  is isotonehomeomorphic to unit interval  $[0, 1]$ . Consequently, each point of  $L$ , not the first element, 1, has an open neighborhood which is homeomorphic to an open interval.

*Proof.* Use transfinite induction.

**PROPOSITION 4.7.**  $L$  is countably compact but is not paracompact, hence is not a compact space.

*Proof.* Let  $A$  be any countably infinite subset of  $L$ . Then,  $A$  will be contained in the union of  $\{I_\alpha : \alpha \in \mathcal{A} \subset W\}$ , where  $W$  is the set of all ordinal numbers less than the first uncountable ordinal, and  $\mathcal{A}$  has at most countably many elements. Let  $\alpha_0$  be the least upper bound of  $\mathcal{A}$ . Then  $[1, \alpha_0]$  is homeomorphic to  $[0, 1]$ . Hence  $A \subset [1, \alpha_0]$ , a compact set, must have a limit point in  $[1, \alpha_0] \subset L$ , so that  $L$  is countably compact. In view of (4.6),  $L$  is locally metrizable [5, p. 80] but  $L$  is not metrizable. By Theorem 2-68 [5, p. 81]  $L$  is not paracompact. Hence, it is not compact.

**PROPOSITION 4.8.** Of any two disjoint closed sets in  $L$ , one is bounded.

Proof is similar [3, 5.12 (b)].

By (4.6), we know that  $L$  is a 1-dimensional manifold with a boundary point 1. Hence we may define the differentiable function on  $L$ .

**PROPOSITION 4.9.** Every function  $f \in C(L)$  is a constant on a tail  $L - L(\alpha)$  where  $\alpha$  depends on  $f$ , and  $L(\alpha) = \{\sigma \in L : \sigma < \alpha\}$ .

Proof is similar to [3, 5.12(c)].

Let  $L^*$  be the union space of  $L$  and the point  $\Omega$ , the first uncountable ordinal. Then,  $L^*$  is a compact 1-dimensional manifold. For each  $f \in C^m(L)$ , we extend  $f$  to a function  $f^*$  on  $L^*$  by defining that  $f^*(\Omega)$  is the final constant value of  $f$ . Evidently  $f^* \in C^m(L^*)$  and is unique. On the other hand, for each  $g \in C^m(L^*)$ ,  $g|L \in C^m(L)$ . Hence  $C^m(L)$  is isomorphic with  $C^m(L^*)$ , under the mapping  $f \mapsto f^*$ .

Since  $L^*$  is compact, every ideal is fixed, and the maximal ideals assume the form  $M_\sigma = \{f^* \in C^m(L^*) : f^*(\sigma) = 0\}$ , where  $\sigma \in L^*$ . By virtue of isomorphism of  $C^m(L^*)$  with  $C^m(L)$ , the maximal ideals in  $C^m(L)$  are in one-one correspondence with those of  $C^m(L^*)$ . Moreover,



the fixed maximal ideals in  $C^m(L)$  correspond to the ideals  $M_\sigma$  in  $C^m(L^*)$  for each  $\sigma \in L$ , leaving just one free maximal ideal in  $C^m(L)$ , namely,  $M_0 = \{f \in C^m(L) : f^* \in M_0\}$ , the one that corresponds to  $M_0$ . Though  $M_0$  is free, it is not hyper-real, for  $C^m(L)/M_0 \cong C^m(L^*)/M \cong \mathbf{R}$ . Hence  $L$  is not  $m$ -realcompact.

**5. Homomorphism,  $a$ -mapping and  $a$ -homeomorphism.** In this section we will describe the relation between any  $a$ -mapping from  $X$  into  $Y$  and homomorphisms from  $\mathfrak{A}'$  to  $\mathfrak{A}$ .

**DEFINITION 5.1.** Let  $X \subset E^{n_1}$ , and  $Y \subset E^{n_2}$ . A mapping  $\tau : X \rightarrow Y$  is said to be a  $C^m$ -mapping at a point  $p$ , if each component of  $\tau(x) = (\tau_1(x_1, \dots, x_{n_1}), \dots, \tau_{n_2}(x_1, \dots, x_{n_1}))$  is  $C^m$  [14, § 3] at  $p$ . If  $\tau$  is  $C^m$  at each point of  $X$ , then  $\tau$  is said to be a  $C^m$ -mapping on  $X$ . If  $\tau$  is a  $C^m$ -mapping, one-one, onto  $Y$  and its inverse mapping,  $\tau^{-1}$ , is also a  $C^m$ -mapping, then  $\tau$  is  $C^m$ -diffeomorphism. We will say then  $X$  and  $Y$  are  $C^m$ -diffeomorphic.

Note that by (5.1),  $X$  and  $Y$  are  $C^m$ -diffeomorphic implies  $n_1 = n_2$ .

**DEFINITION 5.1.'** A mapping  $\tau$  from  $(X, d_1)$  to  $(X, d_2)$  is said to be an  $L_c$ - (resp.  $L$ -) mapping if, for each compact subset  $A$  of  $X$ , there is a positive number  $K_A$  such that  $d_2(\tau(x), \tau(x')) \leq K_A d_1(x, x')$  for all  $x, x' \in A$ . (resp. if there is a positive number  $K$  such that

$$d_2(\tau(x), \tau(x')) \leq Kd(x, x')$$

for all  $x, x' \in X$ ).  $\tau$  is said to be an  $L_c$ - (resp.  $L$ -) homeomorphism, if  $\tau$  is one-one, onto  $Y$  and both  $\tau$  and its inverse  $\tau^{-1}$  are  $L_c$ - (resp.  $L$ -) mappings.

We will use “ $a$ -mapping” to mean  $C^m$ -mapping,  $L_c$ - or  $L$ -mapping, and “ $a$ -homeomorphism” to mean  $C^m$ -diffeomorphism,  $L_c$ - or  $L$ -homeomorphism according as  $\mathfrak{A}$  is  $C^m(X)$ ,  $L_c(X)$  or  $\mathcal{B} = L(X)$ .

**DEFINITION 5.2.** An  $f \in C^m(X)$  is said to be a local  $i$ -th projection at a point  $p$  if there exists a neighborhood  $U$  of  $p$  such that  $f|U = i$ , where  $i$  always denotes the  $i$ -th projection of the space  $E^n$  or  $X \subset E^n$ .

**LEMMA 5.3.** Let  $X$  be any subset of  $E^n$ . For each  $p \in X$  and  $r > 0$ , there are  $h_i, (1 \leq i \leq n), h_i \in C^{\infty*}(X)$  such that  $h_i(x) = x_i$  for all  $x \in \text{cl}_X B_r(p)$ . We call  $h_i, (1 \leq i \leq n)$  the  $i$ -th bounded local projection at  $p$ .

*Proof.* Choose  $r' > r$ . It is well-known that there exists  $g \in C^{\infty*}(E^n)$  such that

$$g(x) = \begin{cases} 1 & \text{if } x \in \text{cl}_{E^n} B_r(p) \\ 0 & \text{if } x \in E^n - B_r(p) \\ 0 < g(x) < 1, & \text{elsewhere.} \end{cases}$$

Set  $h_i(x) = i(x) \cdot g(x)$ .

Let  $C_0^m$  be a subset of  $C^m(Y)$  (resp.  $C^{m*}(Y)$ ), and  $\tau$  be a mapping from  $X$  to  $Y$ . Then we will see what  $C_0^m$  should be in order that  $g \cdot \tau \in C^m(X)$  (resp.  $C^{m*}(X)$ ) for all  $g \in C_0^m$  implies  $\tau$  is a  $C^m$ -mapping from  $X$  into  $Y$ .

**THEOREM 5.4.** *Let  $\tau$  be a mapping from  $X$  to  $Y$  and  $C_0^m$  be a subset of  $C^m(Y)$ .*

(1)  *$\tau$  is a  $C^m$ -mapping implies  $g \cdot \tau \in C^m(X)$  for all  $g \in C_0^m$ .*

(2) *If  $g \cdot \tau \in C^m(X)$  for each  $g \in C_0^m$ , and  $C_0^m$  includes all projections of  $X$ , then  $\tau$  is a  $C^m$ -mapping on  $X$ .*

*Proof.* (1) It is obvious. (2) Since  $g \cdot \tau \in C^m(X)$  for each  $g \in C_0^m$  which includes all projections on  $X$ , we have, in particular,  $i \cdot \tau(x) = \tau_i(x) \in C^m(X)$  for  $1 \leq i \leq n_2$ . Hence, by (5.1)  $\tau$  is a  $C^m$ -mapping.

**THEOREM 5.4.\*** *Let  $\tau$  be a mapping from  $X$  to  $Y$  and  $C_0^m$  be a subset of  $C^{m*}(Y)$ .*

(1)  *$\tau$  is  $C^m$ -mapping implies  $g \cdot \tau \in C^{m*}(X)$  for all  $g \in C_0^m$ ,*

(2) *If  $g \cdot \tau \in C^m(X)$  for each  $g \in C_0^m$ , and  $C_0^m$  includes all local projections, then  $\tau$  is a  $C^m$ -mapping on  $X$ .*

The proof is similar to (5.4)

**THEOREM 5.4.'** *Let  $\tau$  be a mapping from a metric space  $(X, d_1)$  to another metric space  $(Y, d_2)$ .*

(1) *If  $\tau$  is an  $L_c$ -mapping, then  $f \cdot \tau \in L_c(X)$  for all  $f \in L_c(Y)$ .*

(2) *If  $f \cdot \tau \in L_c(X)$  for all  $f \in L_c(Y)$ , then  $\tau$  is an  $L_c$ -mapping of  $(X, d_1)$  into  $(Y, d_2)$ .*

*Proof.* (1) is clear. (2) Consider any compact subset  $A \neq \emptyset$  of  $X$ . We will show that  $\tau$  is an  $L$ -mapping on  $A$ . By [3, 3.8] we know  $\tau$  is continuous. Hence  $\tau[A]$  is compact. Let  $\Phi$  be a mapping from  $L_c(Y)$  to  $L_c(X)$  defined by  $\Phi(f) = f \cdot \tau$  for all  $f \in L_c(Y)$ . Then, it is obvious that  $\Phi$  is a homeomorphism of  $L_c(Y)$  into  $L_c(X)$ . We restrict  $\Phi$  to  $L_c(Y) | \tau[A] = \{f | \tau[A] : f \in L_c(Y)\}$ , then  $\Phi$  is into

$$L_c(X) | A = \{g | A : g \in L_c(X)\}.$$

By compactness of  $A$  and the fact that every function which is Lip-

schitzian on a nonempty subset of a space can be extended to the whole space [7], we can show that  $L_c(X) \mid A = L(A)$ , and

$$L_c(Y) \mid \tau[A] = L(\tau[A]) .$$

By [12, 5.1]  $\tau$  is an  $L$ -mapping on  $A$ . Since  $A$  is arbitrary,  $\tau$  is an  $L_c$ -mapping.

The induced mapping is defined in [3, 10.2]. We are concerned with an  $a$ -mapping  $\tau$  of  $X$  into  $Y$ , where the role of  $D$  in [3, 10.2] is taken by  $E^1$ . The appropriate subset of  $E^{1^Y}$  will be  $\mathfrak{U}'$  or  $\mathfrak{B}'$ . Evidently, the induced mapping  $\tau'$ , defined by  $\tau'(g) = g \cdot \tau \in \mathfrak{A}$  for each  $g \in \mathfrak{U}'$  (resp.  $\mathfrak{B}'$ ) is a homomorphism from  $\mathfrak{U}'$  to  $\mathfrak{A}$  (resp.  $\mathfrak{B}'$  into  $\mathfrak{B}$ ), and  $\tau$  carries the constant functions onto the constant functions identically. Moreover,  $\tau'$  determines the mapping  $\tau$  uniquely.

We now examine the duality relation between  $\tau$  and  $\tau'$ .

**DEFINITION 5.5.** A subset  $A$  of  $X \subset E^n$  is  $C^m$  (resp.  $C^{m*}$ )-embedded in  $X$  if for each  $f \in C^m(A)$  (resp.  $C^{m*}(A)$ ) there is  $g \in C^m(X)$  (resp.  $C^{m*}(X)$ ) such that  $g \mid A = f$ .

**DEFINITION 5.5.'** A subset  $A$  of a metric space  $(X, d)$  is  $L_c$  (resp.  $L_c^*$ , or  $L$ )-embedded in  $X$  if for each  $f \in L_c(A)$  (resp.  $L_c^*(A)$ , or  $L(A)$ ), there is  $g \in L_c(X)$  (resp.  $L_c^*(X)$ , or  $L(X)$ ) such that  $g \mid A = f$ .

We will simply say that a subset  $A$  of a topological space is  $a$  (resp.  $b$ )-embedded if  $A$  is  $C^m$ , or  $L_c$  (resp.  $C^{m*}$ ,  $L_c^*$ , or  $L$ )-embedded.

**THEOREM 5.6.** Let  $\tau$  be an  $a$ -mapping from  $X$  into  $Y$ , and  $\tau'$  be the induced homomorphism  $g \rightarrow g \cdot \tau$  from  $\mathfrak{U}'$  into  $\mathfrak{A}$  (resp.  $\mathfrak{B}'$  into  $\mathfrak{B}$ ).

(1)  $\tau'$  is an isomorphism (into) if and only if  $\tau[X]$  is dense in  $Y$ .

(2)  $\tau'$  is onto if and only if  $\tau$  is an  $a$ -homeomorphism whose image is  $a$  (resp.  $b$ )-embedded.

*Proof.* Having (5.4), (5.4)\*, (5.4)' and (5.3) in hand, the proof is similar to [3, 10.3].

**COROLLARY 5.7.** If  $\tau$  is an  $a$ -homeomorphism from  $X$  onto  $Y$ , then  $\tau'$  is an isomorphism of  $\mathfrak{U}'$  onto  $\mathfrak{A}$ .

**COROLLARY 5.8.** If  $\tau$  is an  $a$ -homeomorphism of a compact space  $X$  to  $Y$ , then the induced mapping  $\tau'$  is onto.

*Proof.* Use the theorem of Whitney's analytic extension [14] and the fact that each Lipschitzian function on a subset can be extended to the whole space [7]. The proof is evident.

Next, we examine the inverse problem of determining when a given homomorphism of  $\mathfrak{A}'$  into  $\mathfrak{A}$  is induced by some  $a$ -mapping from  $X$  into  $Y$ . We shall first consider the homomorphism from  $\mathfrak{A}$  into  $\mathbf{R}$ , i.e., the case in which  $X$  consists of just one point.

**PROPOSITION 5.9.** Any nonzero homomorphism  $\phi$  from  $\mathfrak{A}'$  (or  $\mathfrak{B}'$ ) into  $\mathbf{R}$  is onto  $\mathbf{R}$ . In fact  $\phi(r) = r$  for all  $r \in \mathbf{R}$ .

Proof is similar to [3, 10.5(a)].

**PROPOSITION 5.10.** The correspondence between the homomorphism of  $\mathfrak{A}'$  (or  $\mathfrak{B}'$ ) onto  $\mathbf{R}$ , and the real maximal ideals is one-one.

Proof is similar to [3, 10.5(b)].

**PROPOSITION 5.11.**  $Y$  is  $a$ -realcompact if and only if to each nonzero homomorphism  $\phi$  from  $\mathfrak{A}$  onto  $\mathbf{R}$ , there corresponds a unique point  $y$  of  $Y$  such that  $\phi(g) = g(y)$  for all  $g \in \mathfrak{A}$ .

*Proof.* Use (5.10) and  $a$ -realcompactness.

Our first result about homomorphisms from  $\mathfrak{A}'$  into  $\mathfrak{A}$  for  $X$  is a generalization of (5.11).

**THEOREM 5.12.** Let  $\phi$  be a homomorphism from  $\mathfrak{A}'$  into  $\mathfrak{A}$  such that  $\phi(u) = u$ . If  $Y$  is  $a$ -realcompact, then there exists a unique  $a$ -mapping  $\tau$  of  $X$  into  $Y$  such that  $\tau' = \phi$ .

Notice that the condition  $\phi(u) = u$  is necessary. Proof of the theorem is similar to [3, 10.6].

**COROLLARY 5.13.** An  $a$ -realcompact space  $Y$  contains an image of an  $a$ -mapping of  $X$  if and only if  $\mathfrak{A}$  contains a homomorphic image of  $\mathfrak{A}'$  that included the constant functions on  $X$ .

Proof is similar to [3, 10.9(a)].

**COROLLARY 5.14.** An  $a$ -realcompact space  $Y$  contains an image of an  $a$ -mapping which is dense in  $Y$  if and only if  $\mathfrak{A}$  contains an isomorphic image of  $\mathfrak{A}'$  that includes the constant functions on  $X$ .

Proof is similar to [3, 10.9(b)].

**THE MAIN THEOREM.** *Two  $\alpha$ -realcompact spaces  $X$  and  $Y$  are  $\alpha$ -homeomorphic if and only if  $\mathfrak{A}$  and  $\mathfrak{A}'$  are isomorphic.*

*Proof.* The necessity follows from (5.7).

*Sufficiency.* Let  $\Phi$  be an isomorphism of  $\mathfrak{A}'$  onto  $\mathfrak{A}$ . Then  $\Phi^{-1}$  is an isomorphism of  $\mathfrak{A}$  onto  $\mathfrak{A}'$ . By (5.12), there exist unique  $\alpha$ -mappings  $\tau$  and  $\tau_1$  from  $X$  into  $Y$  and from  $Y$  into  $X$ , respectively, such that  $\Phi(g) = g \cdot \tau$ , and  $\Phi^{-1}(f) = f \cdot \tau_1$ , for each  $g \in \mathfrak{A}'$  and  $f \in \mathfrak{A}$ . Then,  $g(y) = \Phi^{-1}(g \cdot \tau)(y) = (g \cdot \tau) \cdot \tau_1(y) = g \cdot (\tau \cdot \tau_1)(y)$  for all  $y \in Y$ . That is,  $\tau \cdot \tau_1$  is the identity mapping of  $Y$  onto itself. Similarly,  $\tau_1 \cdot \tau$  is the identity mapping of  $X$  onto itself. Thus  $\tau$  and  $\tau_1$  are the inverse mappings of each other. Hence  $X$  and  $Y$  are  $\alpha$ -homeomorphic.

**REMARK.** S.B. Myers [9], L.E. Pursell [11], and M. Nakai [10] have dealt with  $C^m$ -differentiable  $n$ -manifolds. This theorem is applicable to any closed subset of  $E^n$ .

In spite of the remark made in (5.12), every homomorphism is induced, in essence, by an  $\alpha$ -mapping.

**THEOREM 5.16.** *Let  $\Phi$  be a homomorphism from  $\mathfrak{A}'$  (resp.  $\mathcal{B}'$ ) into  $\mathfrak{A}$  (resp.  $\mathcal{B}$ ),  $Y$  be  $\alpha$ -realcompact (resp. compact). Then the set  $E = \{x \in X : \Phi(u)(x) = 1\}$  is open-and-closed in  $X$ . Moreover, there exists a unique  $\alpha$ -mapping  $\tau$  from  $E$  into  $Y$ , such that for any  $g \in \mathfrak{A}'$  (resp.  $\mathcal{B}'$ )  $\Phi(g)(x) = g(\tau(x))$  for all  $x \in E$ , and  $\Phi(g)(x) = 0$  for all  $x \in X - E$ .*

Proof is similar to [3, 10.8].

**COROLLARY 5.17.** *Let  $\Phi$  be a homomorphism from  $\mathfrak{A}'$  into a subring  $R$  of  $\mathfrak{A}$ . If  $Y$  is  $\alpha$ -realcompact, then there exists a unique closed subset  $F$  of  $Y$  such that the kernel of  $\Phi$  is the  $z$ -ideal of all functions in  $R$  that vanish on  $F$ .*

*Proof.* Let  $E = \{x \in X : \Phi(u)(x) = 1\}$ . By (5.16), there exists an  $\alpha$ -mapping  $\tau$  from  $E$  into  $Y$  such that  $\Phi(g)(x) = g(\tau(x))$  for all  $x \in E$ , and  $\Phi(g)(x) = 0$  for all  $x \in X - E$ , for all  $g \in \mathfrak{A}'$ . Let  $F = \text{cl}_Y \tau[E]$ , and  $I = \{g \in \mathfrak{A}' : Z(g) \supseteq F\}$ . We can show easily that  $\ker \Phi = I$ . The uniqueness of  $F$  is clear.

**PROPOSITION 5.18.** *An  $\alpha$ -realcompact (resp. compact) space  $Y$  contains an image of an  $\alpha$ -mapping of  $X$  which is a (resp.  $b$ )-embedded*

if and only if  $\mathfrak{A}$  (resp.  $\mathfrak{B}$ ) is a homomorphism image of  $\mathfrak{A}'$  (resp.  $\mathfrak{B}'$ ).

Proof is similar to [3, 10.9(c)].

REMARK. With the previous results, one can show without any difficulty that the Theorems (2.1), (2.3) (2.4), (2.5) and (2.6) of [6] are true if  $C(X)$  is replaced by  $C^m(X)$  (resp.  $L_c(X)$ ) with the condition that  $\phi(C^m(X))$  contains all projections or all local projections of  $C^m(Y)$  (resp.  $\phi(L_c(X)) = L_c(Y)$ ). However, in Theorem (2.2) [6], only the first three statements are equivalent. For  $\phi$  is not a lattice homomorphism (see [3, 0.5]) from  $C^m(Y)$  to  $C^m(X)$  (resp.  $L_c(Y)$  to  $L_c(X)$ ).

6. Remarks. We have shown that if  $X$  and  $Y$  are two  $a$ -real-compact spaces, then  $\mathfrak{A}$  and  $\mathfrak{A}'$  are isomorphic if and only if  $X$  and  $Y$  are  $a$ -homeomorphic. We shall make some observations about other cases.

Let  $X$  be a subset of  $E^n$ , and  $S_1$  be the set of the projections and the constant functions on  $X$ . Let  $S_2$  be the ring generated by  $S_1$ , and  $\mathcal{R}(X) = \{f/g : f, g \in S_2 \text{ and } Z(g) = \emptyset\}$ . Evidently  $\mathcal{R}(X)$  is a commutative ring of rational functions on  $X$  with unity  $u$  and zero element  $\theta$ . A ring of functions,  $A(X)$ , is said to satisfy property (6-1), if  $\mathcal{R}(X) \subseteq A(X) \subseteq C^m(X)$ , and if  $f \in A(X)$  with  $Z(f) = \emptyset$ , then  $1/f \in A(X)$ .

LEMMA 6.1. *If  $A(X)$  satisfies property (6-1), then there is an  $f \in A(X)$  such that  $f$  belongs to no maximal ideal other than  $M_a = \{f \in A(X) : f(a) = 0\}$  and  $Z(f) = \{a\}$ , for each  $a \in X$ .*

*Proof.* Take  $f(x) = \sum_{i=1}^n (x_i - a_i)^2$ , where  $(a_1, \dots, a_n) = a \in X$ . Then that  $f \in M_a$  and belongs to no other fixed maximal ideal is clear. If  $f$  belongs to a free maximal ideal  $M$ , then there is  $g \in M$  with  $g(a) \neq 0$ . Let  $h = f^2 + g^2$ . We have  $Z(h) = \emptyset$  so that  $1/h \in A(X)$ . Hence  $u = hh^{-1} \in M$ . This is impossible.

LEMMA 6.1'. *For any metric space  $(X, d)$  and  $p \in X$ , there is  $f \in L_c(X)$  such that  $f$  belongs to no maximal ideal other than  $M_p$  and  $Z(f) = \{p\}$ .*

*Proof.* Take  $f(x) = d(p, x)$ . The proof is quite similar to (6.1).

LEMMA 6.2. *If  $A(X)$  and  $A(Y)$  satisfy (6-1), and  $\phi$  is an isomorphism from  $A(X)$  onto  $A(Y)$ , then for any  $M_a \subset A(X)$ ,  $\phi(M_a)$  is a fixed maximal ideal in  $A(Y)$ .*

*Proof.* Consider the image of  $f(x) = \sum_{i=1}^n (x_i - a_i)^2, \Phi(f)$ . We can show that  $Z(\Phi(f)) = \{b\}$  for some  $b \in Y$ . The result follows immediately.

LEMMA 6.2.' *If  $\Phi$  is an isomorphism of  $L_c(X)$  onto  $L_c(Y)$ , then for  $M_p \subset L_c(X)$ ,  $\Phi(M_p)$  is a fixed maximal ideal in  $L_c(Y)$ .*

Proof is similar to (6.2).

LEMMA 6.3. *Let  $B_1$  and  $B_2$  be subrings of  $C(X)$  and  $C(Y)$  respectively, which contain all constant functions,  $\Phi$  be an isomorphism from  $B_2$  to  $B_1$ , and  $X$  be connected. Then  $\Phi$  is the identity on the constant functions.*

*Proof.* It is clear that  $\Phi(r) = r$  for all rational constant functions  $r$ . If  $k$  is an irrational number  $k - r \neq 0$ , for all rational numbers  $r$ . Moreover,  $\Phi(k - r) \cdot \Phi(1/(k - r)) = \Phi(u) = u$ , we have

$$\Phi\left(\frac{1}{k - r}\right) = \frac{1}{\Phi(k) - r}$$

for any rational number  $r$ . Suppose  $\Phi(k)$  is not constant. By continuity of  $\Phi(k)$  and connectedness of  $X$ , we would have  $\Phi(1/(k - r))$  is undefined for some  $r$  and some point of  $X$ . This is a contradiction. Hence  $\Phi(k)$  is constant. By [3, 0.22],  $\Phi$  is the identity on the constant functions.

THEOREM 6.4. *Let  $X$  and  $Y$  be two arbitrary subsets of  $E^n$ . If there are  $A(X)$  and  $A(Y)$  subrings of  $C^m(X)$  and  $C^m(Y)$  satisfying (6-1), and an isomorphism,  $\Phi$ , from  $A(Y)$  onto  $A(X)$  leaving all constant functions unchanged, then  $\Phi$  induces a mapping  $\tau: X \rightarrow Y$  defined by  $\Phi(g) = g \cdot \tau$  and  $\tau$  is a  $C^m$ -diffeomorphism.*

*Proof.* Define  $\tau$  to be a mapping from  $X$  to  $Y$  as follows:  $\tau(x) = \cap Z[\Phi^{-1}(M_x)]$ . By hypothesis and (6.2),  $\Phi^{-1}(M_x)$  is a fixed maximal ideal in  $A(Y)$ . Thus,  $\tau$  is well-defined. Evidently,  $M_{\tau(x)} = \Phi^{-1}(M_x)$ , so that  $\tau$  is one-one. Let  $y_0$  be arbitrary in  $Y$ . Then  $M_{y_0}$  is a fixed maximal ideal in  $A(Y)$ , and  $\Phi(M_{y_0}) = M_{x_0}$  for some  $x_0 \in X$ . Thus  $y_0 = \cap Z[\Phi^{-1}(M_{x_0})] = \tau(x_0)$ . That is,  $\tau$  is onto. Now, for each  $g \in A(Y)$  and each  $x \in X$ , if  $\Phi(g)(x) = r$ , then  $\Phi(g) - r \in M_x$ ,  $g - \Phi^{-1}(r) \in M_{\tau(x)}$ , so that  $g(\tau(x)) = (\Phi^{-1}(r))(\tau(x)) = r(\tau(x)) = r = \Phi(g)(x)$ . Hence  $\Phi(g) = g \cdot \tau$ . Similarly,  $\Phi^{-1}(f) = f \cdot \tau^{-1}$  where  $\tau^{-1}: Y \rightarrow X$ , defined by  $\tau^{-1}(y) = \cap Z[\Phi(M_y)]$ . Since  $f \cdot \tau \in A(X)$  and  $g \cdot \tau^{-1} \in A(Y)$  for each  $g \in A(X)$  and  $f \in A(Y)$ , and  $A(X)$  and  $A(Y)$  contain all projections. By (5.4)  $\tau$  is

$C^m$ -diffeomorphism.

**THEOREM 6.4'.** *Let  $(X, d_1)$  and  $(Y, d_2)$  be any two metric spaces. If there is an isomorphism  $\Phi$  from  $L_c(Y)$  onto  $L_c(X)$  leaving all constant functions unchanged, then  $\Phi$  induces a mapping  $\tau: X \rightarrow Y$  defined by  $\Phi(g) = g \cdot \tau$  and  $\tau$  is an  $L_c$ -homeomorphism.*

Proof is similar to (6.4).

**COROLLARY 6.5.** *Let  $X$  and  $Y$  be two connected subsets of  $E^n$ . If there are subrings  $A(X)$  and  $A(Y)$  of  $C^m(X)$  and  $C^m(Y)$  respectively satisfying (6-1), and an isomorphism,  $\Phi$ , of  $A(Y)$  onto  $A(X)$ , then  $\Phi$  induces a  $C^m$ -diffeomorphism,  $\tau$ , from  $X$  onto  $Y$  such that  $\Phi(g) = g \cdot \tau$  for each  $g \in A(Y)$ .*

*Proof.* Combine (6.3) and (6.4).

**COROLLARY 6.5'.** *Let  $(X, d_1)$  and  $(Y, d_2)$  be two connected metric spaces. If  $\Phi$  be an isomorphism of  $L_c(Y)$  onto  $L_c(X)$ , then  $\Phi$  induces an  $L_c$ -homeomorphism  $\tau$  from  $X$  on  $Y$  such that  $\Phi(g) = g \cdot \tau$  for each  $g \in L_c(Y)$ .*

Proof is similar to (6.5).

**REMARK.** In (6.4) if  $A(X) = \mathcal{R}(X)$ , and  $A(Y) = \mathcal{R}(Y)$ , then  $\tau$  and  $\tau^{-1}$  are not only  $C^m$ , each of their components is a rational function. We may name this mapping as rational-homeomorphism. We know that there is a nonlinear rational-homeomorphism. Let  $X = Y = E^n - (0, \dots, 0)$ , and  $\tau(x) = (\tau_1(x), \dots, \tau_n(x))$  be defined

$$\tau_i(x) = \frac{x_i}{x_1^2 + \dots + x_n^2} \quad \text{for } 1 \leq i \leq n.$$

Then its inverse is known to be  $\tau^{-1}(y) = (\Phi_1(y), \dots, \Phi_n(y))$  with

$$\Phi_j(y) = \frac{y_j}{y_1^2 + \dots + y_n^2}, \quad 1 \leq j \leq n.$$

If the metric spaces are compact subsets of  $E^n$ , then we have the same results as (6.1), (6.2), (6.4) and (6.5) with  $A(X)$  and  $A(Y)$  replaced by  $B(X)$  and  $B(Y)$  respectively, where  $B(X)$  and  $B(Y)$  are the subrings of  $L(X)$  and  $L(Y)$  respectively satisfying the following property:  $\mathcal{R}(X) \subset B(X)$ , and  $f \in B(X)$  with  $Z(f) = \emptyset$  implies  $1/f \in B(X)$ . We know that there is such a proper subring  $B(X)$ . For instance, let  $B_0(X) = \{f \in L(X) : f \in C^3(X)\}$ . Then  $\mathcal{R}(X) \subset B_0(X) \subset L(X)$ .



Next, we will see some algebraic properties of the rings of continuous functions which are inapplicable in the rings of  $C^m$ -differentiable function, where  $1 \leq m \leq \infty$ .

(1) The rings of  $C^m$ -differentiable functions are not lattice-ordered. Let  $X = E^1$ . Consider  $C^m(X)$ . We know  $i(x) = x, i \in C^m(X)$ . But  $|i| \in C^m(X)$ . Thus, neither  $f \wedge \theta$  nor  $f \vee \theta$ , in general, is in  $C^m(X)$ .

(2) We know that in the rings of continuous functions,  $I(f) \geq 0$  if there is  $g \in C(X)$  such that  $g \geq 0$  and  $g \equiv f \pmod{I}$ . (See [3, 5.2 and 5.4(a)].) In the rings of differentiable functions such a  $g$  need not exist. Consider  $X = E^1$ , and  $C^1(X)$ . Let  $I = \{f \in C^1(X) : Z(f) \supset [0, 1]\}$ . Then  $I$  is a  $z$ -ideal, convex, but not absolutely convex. Let  $f_0(x) = x - x^2$ . It is clear that  $f_0 \geq 0$  on a zero-set of  $I$ . But there is no  $g \in C^1(X)$  so that  $g \geq 0$  and  $g$  agrees with  $f_0$  on  $[0, 1]$ .

(3) If  $I$  and  $J$  are  $z$ -ideals in  $C(X)$ , then  $IJ = I \cap J$ . This is not true in  $C^m(X)$ . Let  $X = E^1$ . Consider in  $C^1(E^1)$ ,  $I = J = M_0 = \{f \in C^1(E^1) : f(0) = 0\}$ . Then  $I \cap J = M_0$ . But  $i(x) = x, i \in I \cap J$  and  $i \notin IJ$ . This also shows that the following is not true in  $C^m(X)$  or  $C^{m*}(X)$ . If  $P$  and  $Q$  are prime ideals in  $C$ (or  $C^*$ ) then  $PQ = P \cap Q$ . For  $C^{m*}$ , we take  $X = (-n, n)$ .

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