

Pacific Journal of Mathematics

**A THEOREM OF ROLLE'S TYPE IN E^n FOR FUNCTIONS OF
THE CLASS C^1**

G. P. SZEGŐ

A THEOREM OF ROLLE'S TYPE IN E^n FOR FUNCTIONS OF THE CLASS C^1

G. P. SZEGÖ

In a note recently published W. Leighton presents the following theorem.

THEOREM 1. Let $f(x) = (x_1, \dots, x_n)$ be of class C^2 in E^n . Suppose that $f(x)$ has an isolated relative minimum at $x = 0$, and that $f(0) = 0$. If there is a point $x = 0$ where $f(x) = 0$, then $f(x)$ must have at least one critical point, finite or infinite, in addition to that at the origin.

The proof employed by Prof. Leighton of Theorem (1) is based upon the theory of Morse and for this the condition that $f(x)$ belongs to class C^2 is essential.

In what follows we shall give a proof of Theorem (1) for functions of class C^1 .

This proof will be based upon the so-called "extension theorem" proved in the book by N. P. Bhatia and G. P. Szegö [1, p. 362].

In the sequel, small latin letters will denote vectors (notable exception $t = \text{time}$), small greek letters scalars and capital letters sets.

With E^n we shall denote the n -dimensional euclidean space and with $E^n \cup \{\infty\}$ the n -dimensional euclidean space with the point at infinity adjoined; thus by saying that $y \in E^n \cup \{\infty\}$ is such that $g(y) = 0$ we mean that either exists $y \in E^n$ such that $g(y) = 0$ or that there exists a sequence $\{y^n\} \subset E^n, \|y^n\| \rightarrow \infty$, such that $g(y^n) \rightarrow 0$. With $S(\{y\}, \alpha)$ we shall denote the sphere of radius $\alpha > 0$ with center in the point $y \in E^n$.

Let us first state the extension theorem proved in [1].

(2) **EXTENSION THEOREM.** Let $v = \phi(x)$ and $w = \Psi(x)$ be real-valued functions defined on E^n . Let $M \subset E^n$ be a compact set. Assume that

- (i) $v = \phi(x) \in C^1$
- (ii) $\phi(x) = 0$ for $x \in M$
- (iii) for any sequence $\{x^n\}, \Psi(x^n) \rightarrow 0$ implies $x^n \rightarrow M$.
- (iv) $\Psi(x) = \langle \text{grad } \phi(x), f(x) \rangle$.

Then whatever the local stability properties of M for the system

$$(3) \quad \dot{x} = f(x)$$

may be, these properties are global.

The complete proof and the historical background of this theorem

can be found in [1]. For the case in which $\phi(x) \in C^2$ or for the case in which the solutions of the differential system (3) define a dynamical system, the proof is due to Bhatia and Szegö [2]. The proof for the case $\phi(x) \in C^1$ makes use of a lemma proved by C. Olech [1, Lemma 3. 8.15].

We can then prove the following theorem

THEOREM 4. *Let $v = \phi(x)$ be a real-valued function defined on E^n . Assume that*

(i) $v = \phi(x) \in C^1$

(ii) $\phi(0) = 0$

(iii) $\phi(x) \neq 0, x \neq 0, x \in S(\{0\}, \eta), \eta > 0.$

(iv) $\phi(x^0) = 0, x^0 \neq 0.$

There then exists a point $x^c \in E^n \cup \{\infty\}, x^c \neq 0$, such that

$$(5) \quad \text{grad } \phi(x_c) = 0 .$$

Proof. Consider the differential equation

$$(6) \quad \dot{x} = -\text{grad } \phi(x) ,$$

the real-valued functions $v = \phi(x)$ and

$$(7) \quad \Psi(x) = -\langle \text{grad } \phi(x), \text{grad } \phi(x) \rangle = -(\text{grad } \phi(x))^2 .$$

Suppose that $\text{grad } \phi(x^0) \neq 0$, if not, the theorem is trivially true. Assume that for any sequence $\{x^n\}, \Psi(x^n) \rightarrow 0$ implies $x^n \rightarrow 0$, i.e., that $\text{grad } \phi(x) \neq 0, x \in (E^n \cup \{\infty\}) \setminus \{0\}$. Then from the extension theorem (3) it follows that $\{0\}$ is either globally asymptotically stable or globally completely unstable depending from the sign properties of $\phi(x)$ in $S(\{0\}, \eta)$. On the other hand, if the thesis of the theorem is not true, there must exist a set $M \subset E^n, M \setminus \{x^0 \cup \{0\}\} \neq \emptyset, x^0 \in M$, which is unbounded, such that $\phi(x) = 0$ for all $x \in M$. Assume that $\phi(x)$ is indefinite and changes its sign on M , if not, $\phi(x)$ would be semidefinite and the theorem proved. If $\phi(x)$ changes its sign on M , then there exist $y \in M$, whose stability properties are opposite to the local stability properties of $\{0\}$. For instance if $\{0\}$ is locally asymptotically stable y is unstable and such that $\lim_{t \rightarrow +\infty} x(y, t) \neq 0$, where $x(y, t)$ is a solution of equation (6) with $x(y, 0) = y$. This contradicts the fact that $\{0\}$ is either globally asymptotically stable or globally completely unstable.

In [5] it has been proved that Theorem (4) implies the extension Theorem (2). We have now proved that the Extension Theorem (2) implies, and it is therefore equivalent to, Theorem (4).

REFERENCES

1. N. P. Bhatia and G. P. Szegö, *Dynamical systems: Stability theory and applications*, Lecture Notes in Mathematics, No. 35, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
2. N. P. Bhatia and G. P. Szegö, *An Extension theorem for asymptotic stability*, Differential Equations and Dynamical Systems (J. K. Hale and J. P. LaSalle, Ed.) Academic Press, New York, 1967.
3. W. Leighton, *On Liapunov functions with a single critical point*, Pacific. J. Math. **19** (1966), 467-472.
4. M. Morse, *Relations between the critical points of a real function of n independent variables*, Trans Amer. Math. Soc. **27** (1925), 345-396.
5. G. P. Szegö, *On Global stability properties of nonlinear control systems*, Rept. Contr. NONR-1228(23), 1964.

Received May 10, 1967, and in revised form November 24, 1967. This work has been partially supported by CNR.

UNIVERSITÀ DI MILANO, ITALY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN

Stanford University
Stanford, California

R. R. PHELPS

University of Washington
Seattle, Washington 98105

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

RICHARD ARENS

University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Willard Ellis Baxter, <i>On rings with proper involution</i>	1
Donald John Charles Bures, <i>Tensor products of W^*-algebras</i>	13
James Calvert, <i>Integral inequalities involving second order derivatives</i>	39
Edward Dewey Davis, <i>Further remarks on ideals of the principal class</i>	49
Le Baron O. Ferguson, <i>Uniform approximation by polynomials with integral coefficients I</i>	53
Francis James Flanigan, <i>Algebraic geography: Varieties of structure constants</i>	71
Denis Ragan Floyd, <i>On QF – 1 algebras</i>	81
David Scott Geiger, <i>Closed systems of functions and predicates</i>	95
Delma Joseph Hebert, Jr. and Howard E. Lacey, <i>On supports of regular Borel measures</i>	101
Martin Edward Price, <i>On the variation of the Bernstein polynomials of a function of unbounded variation</i>	119
Louise Arakelian Raphael, <i>On a characterization of infinite complex matrices mapping the space of analytic sequences into itself</i>	123
Louis Jackson Ratliff, Jr., <i>A characterization of analytically unramified semi-local rings and applications</i>	127
S. A. E. Sherif, <i>A Tauberian relation between the Borel and the Lototsky transforms of series</i>	145
Robert C. Sine, <i>Geometric theory of a single Markov operator</i>	155
Armond E. Spencer, <i>Maximal nonnormal chains in finite groups</i>	167
Li Pi Su, <i>Algebraic properties of certain rings of continuous functions</i>	175
G. P. Szegő, <i>A theorem of Rolle's type in E^n for functions of the class C^1</i>	193
Giovanni Viglino, <i>A co-topological application to minimal spaces</i>	197
B. R. Wenner, <i>Dimension on boundaries of ε-spheres</i>	201