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In a note recently published W. Leighton presents the following theorem.

THEOREM 1. **Let $f(x) = (x_1, \cdots, x_n)$ be of class C^2 in E^n. Suppose that $f(x)$ has an isolated relative minimum at $x = 0$, and that $f(0) = 0$. If there is a point $x = 0$ where $f(x) = 0$, then $f(x)$ must have at least one critical point, finite or infinite, in addition to that at the origin.**

The proof employed by Prof. Leighton of Theorem (1) is based upon the theory of Morse and for this the condition that $f(x)$ belongs to class C^2 is essential.

In what follows we shall give a proof of Theorem (1) for functions of class C^1.

This proof will be based upon the so-called "extension theorem" proved in the book by N. P. Bhatia and G. P. Szegö [1, p. 362].

In the sequel, small latin letters will denote vectors (notable exception t = time), small greek letters scalars and capital letters sets.

With E^n we shall denote the n-dimensional euclidean space and with $E^n \cup \{\infty\}$ the n-dimensional euclidean space with the point at infinity adjoined; thus by saying that $y \in E^n \cup \{\infty\}$ is such that $g(y) = 0$ we mean that either exists $y \in E^n$ such that $g(y) = 0$ or that there exists a sequence $\{y^n\} \subset E^n$, $\| y^n \| \to \infty$, such that $g(y^n) \to 0$. With $S(\{y\}, \alpha)$ we shall denote the sphere of radius $\alpha > 0$ with center in the point $y \in E^n$.

Let us first state the extension theorem proved in [1].

(2) EXTENSION THEOREM. *Let $v = \phi(x)$ and $w = \Psi(x)$ be real-valued functions defined on E^n. Let $M \subset E^n$ be a compact set. Assume that*

(i) $v = \phi(x) \in C^1$

(ii) $\phi(x) = 0$ *for* $x \in M$

(iii) *for any sequence* $\{x^n\}$, $\Psi(x^n) \to 0$ *implies* $x^n \to M$.

(iv) $\Psi(x) = \langle \text{grad}\, \phi(x), f(x) \rangle$.

Then whatever the local stability properties of M for the system

$$\dot{x} = f(x) \tag{3}$$

may be, these properties are global.

The complete proof and the historical background of this theorem can be found in [1]. For the case in which $\phi(x) \in C^2$ or for the case in which the solutions of the differential system (3) define a dynamical system, the proof is due to Bhatia and Szegö [2]. The proof for the case $\phi(x) \in C^1$ makes use of a lemma proved by C. Olech [1, Lemma 3. 8.15].

We can then prove the following theorem

THEOREM 4. *Let $v = \phi(x)$ be a real-valued function defined on E^n. Assume that*

(i) $v = \phi(x) \in C^1$

(ii) $\phi(0) = 0$

(iii) $\phi(x) \neq 0$, $x \neq 0$, $x \in S(\{0\}, \eta)$, $\eta > 0$.

(iv) $\phi(x^0) = 0$, $x^0 \neq 0$.

There then exists a point $x^c \in E^n \cup \{\infty\}$, $x^c \neq 0$, such that

$$\text{grad } \phi(x_c) = 0 \; . \tag{5}$$

Proof. Consider the differential equation

$$\dot{x} = -\text{grad}\, \phi(x) \; , \tag{6}$$

the real-valued functions $v = \phi(x)$ and

$$\Psi(x) = - \langle \text{grad}\, \phi(x), \text{grad}\, \phi(x) \rangle = - (\text{grad}\, \phi(x))^2 \; . \tag{7}$$

Suppose that grad $\phi(x^0) \neq 0$, if not, the theorem is trivially true. Assume that for any sequence $\{x^n\}$, $\Psi(x^n) \to 0$ implies $x^n \to 0$, i.e., that grad $\phi(x) \neq 0$, $x \in (E^n \cup \{\infty\}) \backslash \{0\}$. Then from the extension theorem (3) it follows that $\{0\}$ is either globally asymptotically stable or globally completely unstable depending from the sign properties of $\phi(x)$ in $S(\{0\}, \eta)$. On the other hand, if the thesis of the theorem is not true, there must exist a set $M \subset E^n$, $M \backslash \{x^0 \cup \{0\}\} \neq \phi$, $x^0 \in M$, which is unbounded, such that $\phi(x) = 0$ for all $x \in M$. Assume that $\phi(x)$ is indefinite and changes its sign on M, if not, $\phi(x)$ would be semidefinite and the theorem proved. If $\phi(x)$ changes its sign on M, then there exist $y \in M$, whose stability properties are opposite to the local stability properties of $\{0\}$. For instance if $\{0\}$ is locally asymptotically stable y is unstable and such that $\lim_{t \to +\infty} x(y, t) \neq 0$, where $x(y, t)$ is a solution of equation (6) with $x(y, 0) = y$. This contradicts the fact that $\{0\}$ is either globally asymptotically stable or globally completely unstable.

In [5] it has been proved that Theorem (4) implies the extension Theorem (2). We have now proved that the Extension Theorem (2) implies, and it is therefore equivalent to, Theorem (4).

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