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A CO-TOPOLOGICAL APPLICATION TO MINIMAL SPACES

GIOVANNI VIGLINO

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A space (X, τ) which satisfies a topological property P is said to be minimal- P if $T = \{\tau' \mid \tau' \text{ is a } P\text{-topology on } X; \tau' \not\leq \tau\} = \emptyset$. For example, a Hausdorff space (X, τ) is minimal Hausdorff if there exists no Hausdorff topology on X which is strictly weaker than τ . The purpose of this paper is to show that for certain properties one need only consider a subset of T "induced" by τ to determine if (X, τ) is minimal- P .

Notation. Let β be an open base for the space (X, τ) . τ_β will denote the topology on X generated by the subbase $\{X \setminus Cl_\tau B \mid B \in \beta\}$.

REMARK. J. de Groot in his investigation for a general classification of Baire spaces considered the above topologies (cf. [1], [4]). These topologies have come to be known as co-topologies.

DEFINITIONS. A filter base is regular if it is open and equivalent to a closed filter base.

A filter base \mathcal{U} is Urysohn if for each nonadherent point a , there exists a neighborhood V and $G \in \mathcal{U}$ such that $Cl_\tau V \cap Cl_\tau G = \emptyset$.

REMARK. In this paper, the Bourbaki convention for the topological separation properties will be observed; specifically, all spaces are assumed to be Hausdorff.

The proof of the following lemmas are left to the reader. A proof for the regular case of Lemma 1 is similar to the proof of Theorem 2 in [3].

LEMMA 1. *Let (X, τ) be a Hausdorff (Urysohn; regular) space; let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a nonconvergent open (Urysohn; regular) filter base with unique adherent point x_0 ; let $\beta = \mathcal{N} \cup \mathcal{M}$ where*

$$\mathcal{N} = \{N \mid N \in \tau \text{ and } x_0 \in Cl_\tau N\}$$

and

$$\mathcal{M} = \{M \mid M \in \tau \text{ and } M \subset X \setminus Cl_\tau U_\alpha \text{ for some } \alpha \in A\}.$$

Then (i) β is a base for τ ; and
 (ii) τ_β is a Hausdorff (Urysohn; regular) topology strictly

weaker than τ .

LEMMA 2. *Let (X, τ) be a normal (completely normal) space; let \mathcal{Z} be a nonconvergent regular filter base with unique adherent point x_0 ; let β be defined as in Lemma 1. Then τ_β is a normal (completely normal) topology strictly weaker than τ .*

In the following theorem P denotes any of the following properties: (i) Hausdorff, (ii) Urysohn, (iii) regular, (iv) completely regular, (v) normal, (vi) completely normal, (vii) locally compact. In [2], [3], [5] it is shown that there exist minimal Hausdorff, minimal Urysohn, and minimal regular spaces which are not compact, while for properties (iv) through (vii) minimal- P is equivalent to compactness.

THEOREM. *A P -space (X, τ) is minimal- P if and only if $\{\tau_\beta \mid \tau_\beta \text{ is } P; \tau_\beta \neq \tau\} = \emptyset$.¹*

Proof. Necessity, in each case, follows from the fact that $\tau_\beta \leq \tau$ for every open base β .

Sufficiency for property (i) through (iii): Suppose (X, τ) is not minimal Hausdorff (Urysohn; regular). Then there exists an open (Urysohn; regular) filter base $\mathcal{Z} = \{U_\alpha\}_{\alpha \in A}$ with unique adherent point x_0 , which does not converge (see [5], [2]). By Lemma 1, there exists a base β for τ such that $\tau_\beta \not\leq \tau$ and τ_β is Hausdorff (Urysohn; regular).

Sufficiency for completely regular²: Suppose (X, τ) is not compact.

Let (Y, τ') denote a compact extension of (X, τ) . Take and fix $p \in Y \setminus X$. Let \mathcal{S} be the filter base of open neighborhoods of p , and \mathcal{S}^* denote the trace of \mathcal{S} in X . Considered as a filter base in (Y, τ') , \mathcal{S}^* has a unique adherent point, namely p . Thus \mathcal{S}^* has no adherent point in (X, τ) . Fix an element x_0 in X . Let $\beta = \mathcal{N} \cup \mathcal{M}$ where $\mathcal{N} = \{N \mid N \in \tau \text{ and } x_0 \in Cl_\tau N\}$ and $\mathcal{M} = \{M \mid M \in \tau \text{ and } M \subset X \setminus Cl_\tau S^* \text{ for some } S^* \in \mathcal{S}^*\}$. One can show β is an open base for τ . Similarly one can show that $\mathcal{H} = \{X/Cl_\tau H \mid H \in \mathcal{N} \cup \mathcal{M}\}$ is a base for τ_β .

We will now show that $\tau_\beta \neq \tau$ and (X, τ_β) is completely regular. Let us first note that since (X, τ) is regular and since $\mathcal{N} \subset \beta$, then $G \in \tau_\beta$ whenever $G \in \tau$ and $x_0 \notin G$. Hence if f is continuous on (X, τ) then f is continuous everywhere on (X, τ_β) except possibly at x_0 . Now there exists $S^* \in \mathcal{S}^*$ such that $x_0 \in Cl_\tau S^*$. Since τ is regular, then there exists $U \in \tau$ such that $x_0 \in U$ and $Cl_\tau U \cap Cl_\tau S^* = \emptyset$. Since any element of τ_β which contains x_0 must meet S^* , then $U \notin \tau_\beta$. Thus

¹ The result for $p = \text{Hausdorff}$ was independently obtained by G. Strecker.

² The technique used by Berri in [2] to show that a space is compact if it is minimal completely regular is extensively used in this proof.

$\tau_\beta \neq \tau$.

We complete the proof by showing τ_β is completely regular. Take $b \in X$ and $X \setminus Cl_\tau H \in \mathcal{K}$ where $b \notin Cl_\tau H$ and $H \in \mathcal{N} \cup \mathcal{M}$. We wish to show there exists a continuous, real-valued function f on (X, τ_β) , such that $f(b) = 1$ and $f(x) = 0$ for all $x \in Cl_\tau H$. Suppose $H \in \mathcal{N}$. Then $x_0 \in Cl_\tau H$. Let $S^* \in \mathcal{S}^*$ be such that $b \notin Cl_\tau S^*$; Since (X, τ) is regular, then there exists $V \in \tau$ such that $b \in V$ and

$$Cl_\tau V \cap Cl(H \cup S^*) = \emptyset .$$

Since (X, τ) is completely regular, then there exists a continuous, real-valued function f such that $f(b) = 1$ and $f(x) = 0$ for all $x \in X \setminus V$. By a previous remark, f is continuous at every point of (X, τ_β) except possibly at $x = x_0$. We will now show f is continuous at $x = x_0$. Now for all $x \in X \setminus V$, $f(x) = 0$. Since $Cl_\tau V \cap Cl_\tau(H \cup S^*) = \emptyset$, then $f(x) = 0$ for all $x \in X \setminus Cl_\tau V$. Thus f is continuous at all $x \in X \setminus Cl_\tau V$, and hence at all $x \in Cl_\tau(H \cup S^*)$. Therefore f is continuous at x_0 .

Similarly one can show that if $H \in \mathcal{M}$, then there exists a real-valued continuous function f on (X, τ_β) such that $f(b) = 1$ and $f(x) = 0$ for each $x \in Cl_\tau H$.

Sufficiency for properties (v) and (vi): Suppose the normal (completely normal) space (X, τ) is not compact. Then X is not minimal regular since a minimal regular normal (completely normal) space is minimal completely regular. Hence there exists a nonconvergent regular filter base \mathcal{U} with a unique adherent point x_0 . By Lemma 2, there exists a base β for τ such that $\tau_\beta \not\leq \tau$ and τ_β is normal (completely normal).

Sufficiency for locally compact: Suppose (X, τ) is not minimal locally compact (i.e., not compact). Let (Y, τ') denote the Alexandroff compactification of X with $Y = X \cup \{p\}$ where $p \notin X$. Fix an element x_0 in X and construct $\beta = \mathcal{N} \cup \mathcal{M}$ as in the proof of sufficiency for completely regular spaces. One can show $\tau_\beta \not\leq \tau$ and τ_β is locally compact, and in fact, compact.

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