A NOTE ON CLT GROUPS

HENRY GILBERT BRAY
A NOTE ON CLT GROUPS

HENRY G. BRAY

Let $A, B, C$ be respectively the class of all finite supersolvable groups, the class of all finite groups which satisfy the converse to Lagrange's theorem, and the class of all finite solvable groups. We show that $A \subseteq B \subseteq C$, and give examples to show that both of the inclusions are actually proper.

Throughout, ‘$n$’, ‘$t$’, ‘$a_1$’, ‘$a_2$’, ⋯, ‘$a_t$’ will denote positive integers; ‘$p_1$’, ‘$p_2$’, ⋯, ‘$p_t$’ will denote pairwise distinct positive integer primes. If $G$ and $H$ are finite groups, then, ‘$G'$’ will denote the commutator subgroup of $G$, ‘$G \times H$’ will denote the external direct product of $G$ and $H$, and ‘$|G|$’ will denote the order of $G$. ‘$A_n$’ will denote the alternating group on 4 symbols, ‘$e$’ will denote the identity of $A_n$, and ‘$C_2$’ will denote the cyclic group of order 2.

We are concerned here only with finite groups; throughout, when we say ‘group’, we intend this to be read as ‘finite group’, and ‘$G$’ will always denote a finite group. Our version of the converse to Lagrange's theorem is as follows:

DEFINITION. $G$ is a CLT group if and only if for each $d$, the following holds: if $d$ is a positive integer divisor of $|G|$, then $G$ has at least one subgroup $H$ with $|H| = d$.

All terminology not used in the above definition will be that of [2].

**Lemma 1.** $|G| = n = p_1^{n_1}p_2^{n_2} ⋯ p_t^{n_t}$ and $n_i = n/p_i^{n_i}$ for $i = 1, 2, ⋯, t$. Then $G$ is solvable if and only if $G$ has subgroups with orders $n_1, n_2, ⋯, n_t$.

**Proof.** This follows readily from Theorem 9.3.1, p. 141, and Theorem 9.3.3, p. 144 of [2].

**Lemma 2.** $|G| = n = p_1^{n_1}p_2^{n_2} ⋯ p_t^{n_t}$ with $p_1 < p_2 < ⋯ < p_t$. Then if $G$ is supersolvable, $G$ has normal subgroups with orders 1, $p_i$, $p_i^2$, ⋯, $p_i^t$.

**Proof.** This follows readily from Corollary 10.5.2, p. 159 of [2].

**Theorem 1.** Every CLT group is solvable.

**Proof.** This is trivial if $|G| = 1$. Let $G$ be a CLT group with $|G| = n = p_1^{n_1}p_2^{n_2} ⋯ p_t^{n_t}$, and let $n_i = n/p_i^{n_i}$ for $i = 1, 2, ⋯, t$; since

229
each $n_i$ is a divisor of $|G|$, $G$ must have subgroups with orders $n_1, n_2, \ldots, n_t$. Applying Lemma 1, we conclude that $G$ is solvable, and this completes our proof.

The author wishes to thank Professor M. Hall for pointing out the proof of Theorem 1.

**Theorem 2.** Every supersolvable group is CLT.

**Proof.** This is trivial if $|G| = 1$. We shall use induction on the number of positive integer primes dividing $|G|$ if $|G| > 1$.

If $G$ is any group with $|G| = p_1^{a_1}$, then Sylow's theorem tells us that $G$ is CLT; in fact, any finite $p_i$-group is supersolvable, but we do not need this.

Suppose now that every supersolvable group whose order is divisible by exactly $t$ distinct positive integer primes is CLT, and let $G$ be a supersolvable group with $|G| = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}p_{t+1}^{a_{t+1}}$, $p_1 < p_2 < \cdots < p_t < p_{t+1}$. We shall show that $G$ is CLT, and our conclusion will follow. Let $d$ be a positive integer divisor of $|G|$; we wish to show that $G$ has a subgroup of order $d$. We may write $d = p_1^{b_1}p_2^{b_2} \cdots p_t^{b_t}p_{t+1}^{b_{t+1}} = r^{b_1}p_1^{b_1}p_2^{b_2} \cdots p_{t+1}^{b_{t+1}}$, where $b_i$ is an integer and $0 \leq b_i \leq a_i$ for each $i = 1, 2, \ldots, t, t+1$, and $r = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$. Since $G$ is supersolvable, $G$ is solvable, and we may apply Lemma 1 to conclude that $G$ has a subgroup $H$ with $|H| = n_{t+1} = p_1^{a_1}p_2^{a_2} \cdots p_t^{a_t}$. Now, $H$ is a subgroup of $G$, and $G$ is supersolvable; hence, $H$ is supersolvable and $|H|$ is divisible by exactly $t$ distinct positive integer primes. By our induction hypothesis, $H$ is CLT; since $r$ is a divisor of $n_{t+1}$ and $n_{t+1} = |H|$, it follows that $H$ must have a subgroup $R$ with $|R| = r$. Thus, $R$ is a subgroup of $G$ with $|R| = r$; since $G$ is supersolvable and $p_{t+1}$ is the largest prime dividing $|G|$, we may apply Lemma 2 to conclude that $G$ has a normal subgroup $P$ with $|P| = p_{t+1}^{b_{t+1}}$. Now let $RP$ be the set of all products $xy$ with $x \in R$ and $y \in P$; since $P$ is a normal subgroup of $G$, $RP$ is a subgroup of $G$. Also, $|R|$ and $|P|$ are relatively prime, so that $|RP| = |R| \cdot |P|/|R \cap P| = |R| \cdot |P|$; hence, $RP$ is a subgroup of $G$ with $|RP| = |R| \cdot |P| = rp_{t+1}^{b_{t+1}} = d$, and this completes our proof.

**Remark.** Since every subgroup of a supersolvable group is supersolvable, it is clear that Theorem 2 can be used to prove the following: If $G$ is supersolvable, then every subgroup of $G$ is CLT. Sometime after the author had obtained Theorem 2, he became aware of the following (due to Professor W. Deskins): $G$ is supersolvable if and only if every subgroup of $G$ (including $G$ itself) is CLT. This appears in [1].

**Lemma 3.** Let $H$ be any group with $|H| = h$, where $h$ is odd.
Then \( |A_4 \times H| = 12h \), and \( A_4 \times H \) has no subgroups of order 6h.

Proof. Suppose to the contrary that \( A_4 \times H \) has a subgroup \( K \) with \( |K| = 6h \); then \( K \) has index 2 in \( A_4 \times H \), so that \( K \) is a normal subgroup of \( A_4 \times H \) and \( |(A_4 \times H)/K| = 2 \). Hence, \((A_4 \times H)/K\) is Abelian, so that \((A_4 \times H)/K\)' is a subgroup of \( K \); it follows that \(|A_4'|\) must divide \(|K|\). Now \( A_4' = \{e, (12)(34), (13)(24), (14)(23)\} \), so that 4 must divide \(|K| = 6h\); this is not possible, since \( h \) is odd, and this completes our proof.

**Lemma 4.** Let \( H \) be any group of odd order; then \( A_4 \times H \) is solvable and not CLT.

Proof. According to Thompson and Feit, \( H \) is solvable; since \( A_4 \) is solvable, it follows that \( A_4 \times H \) is solvable. The result of Lemma 3 shows that \( A_4 \times H \) is not CLT, and this completes our proof.

**Lemma 5.** Let \( G \) be any CLT group; then \((A_4 \times C_2) \times G\) is CLT and not supersolvable.

Proof. It is clear that a finite direct product of CLT groups is itself CLT, and it is clear that \( A_4 \times C_2 \) is CLT; it follows that \((A_4 \times C_2) \times G\) is CLT. Now Lemma 3 shows that \( A_4 \) is not CLT, and Theorem 2 then shows that \( A_4 \) is not supersolvable; it follows that \((A_4 \times C_2) \times G\) is not supersolvable, and this completes our proof.

Our results show that the class of CLT groups fits properly between the class of supersolvable groups and the class of solvable groups. As a closing remark, we note the following: If \( G \) is supersolvable (solvable) then every subgroup of \( G \) and every factor group of \( G \) is supersolvable (solvable); that this is not true for CLT groups in general is shown by the following example. Let \( M \) be any CLT group; then \((A_4 \times C_2) \times M\) is CLT, but \((A_4 \times C_2) \times M\) has \( A_4 \) as both a subgroup and a factor group, and \( A_4 \) is not CLT.

**References**


Received November 6, 1967.

San Diego State College
San Diego, California
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsu sha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.
Leonard E. Baum and George Roger Sell, Growth transformations for functions on manifolds .................................................. 211
Henry Gilbert Bray, A note on CLT groups ................................................. 229
Douglas Napier Clark, On matrices associated with generalized interpolation problems .................................................. 241
Richard Brian Darst and Euline Irwin Green, On a Radon-Nikodym theorem for finitely additive set functions .......................... 255
Carl Louis DeVito, A note on Eberlein’s theorem ........................................ 261
P. H. Doyle, III and John Gilbert Hocking, Proving that wild cells exist ...... 265
Leslie C. Glaser, Uncountably many almost polyhedral wild \((k - 2)\)-cells in \(E^k\) for \(k \geq 4\) ........................................ 267
Samuel Irving Goldberg, Totally geodesic hypersurfaces of Kaehler manifolds ........................................................................ 275
Donald Goldsmith, On the multiplicative properties of arithmetic functions .......................................................... 283
Jack D. Gray, Local analytic extensions of the resolvent ......................... 305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, Commutativity theorems for nonassociative rings with a finite division ring homomorphic image ........................................ 325
André (Piotrowsky) De Korvin, Normal expectations in von Neumann algebras ..................................................................... 333
James Donald Kuelbs, A linear transformation theorem for analytic Feynman integrals ..................................................... 339
W. Kuich, Quasi-block-stochastic matrices ............................................ 353
Richard G. Levin, On commutative, nonpotent archimedean semigroups ........................................................................ 365
James R. McLaughlin, Functions represented by Rademacher series ........ 373
Calvin R. Putnam, Singular integrals and positive kernels ....................... 379
Harold G. Rutherford, II, Characterizing primes in some noncommutative rings ........................................................................ 387
Benjamin L. Schwartz, On interchange graphs ....................................... 393
Satish Shirali, On the Jordan structure of complex Banach\(^*\)algebras .......... 397
Earl J. Taft, A counter-example to a fixed point conjecture ....................... 405
J. Roger Teller, On abelian pseudo lattice ordered groups ...................... 411