Pacific Journal of Mathematics

ON A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

RICHARD BRIAN DARST AND EULINE IRWIN GREEN

Vol. 27, No. 2 February 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 2, 1968

ON A RADON-NIKODYM THEOREM FOR FINITELY ADDITIVE SET FUNCTIONS

R. B. DARST and EULINE GREEN

The purpose of this note is to comment on and extend recent results of C. Fefferman. A proof of his Radon-Nikodym theorem that is, perhaps, more amenable to generalization is given. A Lebesgue decomposition is also obtained.

Since the notations in [3] and [7] conflict, we shall make the following compromises in notation and terminology, and beg the reader's indulgence.

Let S be a set, Σ be an algebra of subsets of S, C be the complex numbers, and R be the real numbers. Let $H(C)=H(S, \Sigma: C)$ denote the set of all bounded, complex valued, finitely additive set functions on Σ. Then $H(R)$ will denote the real valued elements of $H(C)$. If $\alpha \in H(C)$ and $E \in \Sigma$, we denote the total variation of α over E by $v(\alpha, E)$. If $\alpha, \beta \in H(C)$ then

(i) α is absolutely continuous with respect to $\beta (\alpha \ll \beta)$ means: given $\varepsilon > 0$, there exists $\delta > 0$ such that $v(\beta, E) < \delta$ $(E \in \Sigma)$ implies $v(\alpha, E) < \varepsilon$.

(ii) α is singular with respect to $\beta (\alpha \perp \beta)$ means: given $\varepsilon > 0$, there exists $E \in \Sigma$ such that $v(\alpha, E) < \varepsilon$ and $v(\beta, S\text{-}E) < \varepsilon$.

The classical Radon-Nikodym theorem (eg., [6, Th. III. 10.2]) asserts that if Σ is a sigma algebra and λ is a countably additive element of $H(C)$, then λ can be given an integral representation with respect to a nonnegative, countably additive element μ of $H(R)$ if, and only if, λ is absolutely continuous with respect to μ.

In 1939, S. Bochner published a generalization ([1]) which removed the restrictions that Σ be a sigma algebra and that the set functions be countably additive. Then S. Bochner and R. S. Phillips [2] used a vector lattice approach to give a new proof of Bochner's Theorem and, also, to obtain a Lebesgue decomposition. S. Leader [8] studied the L^p-spaces associated with finitely additive measures. A representation for the case where $\mu \in H(R)$ appeared ([3]) in 1962. Theorem III. 10.7 of [6] supplements the classical theorem by allowing μ to be complex valued, and recently C. Fefferman ([7]) extended the latter result to the case of a general algebra of subsets of a set.

Let us turn to some comments on the paper of Fefferman.

(i) The definition of absolute continuity given in [7] seems to contain a misprint: Suppose that $S=[-1, 1]$, Σ is the sigma algebra of Lebesgue measurable subsets of S, α is Lebesgue measure m res-

tricted to $[-1, 0]$ (i.e., $\alpha(E)=m(E\cap[-1, 0])$), β is Lebesgue measure restricted to $[0, 1]$, $\mu=\alpha-\beta$, $\gamma=\alpha+\beta$ and $E=S$. Then $\mu(E)=0$ while $\gamma(E)=2$ and, hence, γ is not absolutely continuous with respect to μ, according to [7]. Replacing $|\mu(E)|$ by $v(\mu, E)$ or by sup $\{|\mu(F)|; F\in\Sigma, F\subset E\}$ in the definition of absolute continuity given in [7] prevents the pathology illustrated by the preceding example.

(ii) After correcting the misprint, one should replace the last statement in [7, p. 35, paragraph 1] by "Unless γ is bounded and countably additive the last two definitions need not be equivalent." (See [4, Th. 3.4]. For a counterexample to the original statement, let S be the positive integers, let Σ be the σ-algebra of all subsets of S, let $\mu(E)=\Sigma_{n\in E}2^{-n}$, and let γ be a Banach measure).

(iii) Existence of a Lebesgue decomposition in the setting of [7] follows immediately from [5] upon letting $T_\varepsilon=\{E\in\Sigma;\ v(\mu, E)<\varepsilon\}$.

(iv) The first four lemmas in [7] are obtained in [3] by a technical modification of a standard proof of the classical Radon-Nikodym theorem.

(v) While the last two lemmas in [7] are neat, our argument seems to make certain generalizations more transparent. Consider, for example, finitely additive set functions on Σ to a Banach space over R with basis. The coordinate functionals are finitely additive, and if enough conditions are imposed, then our procedure can be applied.

For the sake of completeness and clarity, we state Theorem 2.2 of [3].

If $f, g\in H(R)$, then there exist uniquely functions h and s in $H(R)$ such that

(i) $h\ll g$,

(ii) $s\perp g$,

(iii) $f=h+s$

(iv) $E\in\Sigma$ implies $v(f, E)=v(h, E)+v(s, E)$.

Moreover, there exists a sequence $\{y_n\}$ of $(S, \Sigma: R)$-simple functions which converges in g-measure and such that if $h_n(E)=\int_E y_n dg$ for each $E\in\Sigma$, then $\lim_n v(h-h_n, S)=0$.

The following is the desired extension.

THEOREM 1. *Suppose $\alpha, \beta\in H(C)$ and $\alpha\ll\beta$. Then there exists a sequence $\{z_n\}$ of $(S, \Sigma: C)$-simple functions such that if $\alpha_n(E)=\int_E z_n d\beta$ for each $E\in\Sigma$, then $\lim_n v(\alpha-\alpha_n, S)=0$.*

In order to apply Theorem 2.2, we need the following

LEMMA 1. *Suppose*

(i) $\alpha, \beta, \gamma \in H(C)$,

(ii) $\{g_n\}$ *is a sequence of* $(S, \Sigma: C)$*-simple functions such that, if* $\gamma_n(E) = \int_E g_n d\alpha$ *for each* $E \in \Sigma$, *then* $\lim_n v(\gamma - \gamma_n, S) = 0$,

(iii) $\beta \perp \gamma$.

Then there exists a sequence $\{h_n\}$ *of* $(S, \Sigma: C)$*-simple functions such that if* $\gamma'_n(E) = \int_E h_n d\alpha$ and $\gamma''_n(E) = \int_E h_n\, d(\alpha + \beta)$ *for each* $E \in \Sigma$, *then* $\lim_n v(\gamma - \gamma'_n, S) = \lim_n v(\gamma - \gamma''_n, S) = 0$.

Proof. Let $M_n = \max\{|\, g_n(x) \mid x \in S\}$. Since $\gamma \perp \beta$, we choose a sequence $\{A_n\}$ of elements of Σ such that $v(\beta, A_n) < a_n$ and $v(\gamma, S - A_n) < a_n$ where $a_n = \min\{1/nM_n, 1/n\}$. Define $\{h_n\}$ by $h_n(x) = g_n(x)$ if $x \in A_n$ and $h_n(x) = 0$ if $x \in S - A_n$. Let $\varepsilon > 0$ be given and choose N so large that $n \geqq N$ implies both $v(\gamma - \gamma_n, S) < \varepsilon/3$ and $1/n < \varepsilon/3$. Then if $E \in \Sigma$ and

$$n \geq N, |\, \gamma(E) - \gamma'_n(E) \,| \leqq |\, \gamma(E) - \gamma_n(E) \,| + |\gamma_n(E) - \gamma'_n(E) - \gamma(E \cap (S - A_n)) \,| + |\, \gamma(E \cap (S - A_n)) \,| < (\varepsilon/3) + |\, \gamma_n(E \cap (S - A_n)) - \gamma(E \cap (S - A_n)) \,| + v(\gamma, S - A_n) < (\varepsilon/3) + (\varepsilon/3) + (\varepsilon/3) = \varepsilon\,.$$

Therefore, $\lim_n v(\gamma - \gamma'_n, S) = 0$. Finally, for $E \in \Sigma$ and

$$n \geq N, |\, \gamma(E) - \gamma''_n(E) \,| \leqq |\, \gamma(E) - \gamma'_n(E) \,| + \left|\int_E h_n d\beta\right| < \varepsilon + \left|\int_{E \cap A_n} g_n d\beta\right| \leqq \varepsilon + M_n V(\beta, A_n) < \varepsilon + M_n \frac{1}{nM_n} < \varepsilon + \frac{\varepsilon}{3} < 2\varepsilon\,.$$

Therefore $\lim_n v(\gamma - \gamma''_n, S) = 0$, and Lemma 1 is proved.

Proof of Theorem 1. Any $\alpha \in H(C)$ may be separated into its real and complex parts, so it suffices to prove the theorem when α is real valued. According to Theorem 2.2, we can express β as follows: $\beta = \mu + i\nu = (\mu_1 + \mu_2) + i(\nu_1 + \nu_2)$ where $\mu, \nu, \mu_1, \mu_2, \nu_1, \nu_2 \in H(R)$ and

(1) (a) $\mu_1 \perp \nu$ (b) $\mu_2 \ll \nu$

(c) $\nu_1 \perp \mu$ (d) $\nu_2 \ll \mu$.

We also write $\alpha = \alpha_1 + \alpha_2$ where

(2) (a) $\alpha_1 \perp \nu$ (b) $\alpha_2 \ll \nu$.

Notice now that $\alpha_1 \ll \mu_1$. The reason for this is:

$\alpha_1 \ll \alpha \ll \beta$ and (2) (a) imply $\alpha_1 \ll \mu = \mu_1 + \mu_2$. But (1) (b) and (2) (a) imply $\alpha_1 \perp \mu_2$, and hence $\alpha_1 \ll \mu_1$. Therefore, again by Theorem 2.2, there exists a sequence $\{f_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E f_n d\mu_1 = \alpha_1(E)$ uniformly for $E \in \Sigma$. Since $\alpha_1 \perp \nu$ and $\alpha_1 \perp \mu_2$, we conclude that $\alpha_1 \perp \mu_2 + i\nu$, and apply Lemma 1 to get a sequence $\{f'_n\}$ of $(S, \Sigma: C)$-simple functions such that

$$\text{(A)} \qquad \lim_n \int_E f'_n d(\mu_1 + (\mu_2 + i\nu)) = \lim_n \int_E f'_n d\beta = \alpha_1(E)$$

uniformly for $E \in \Sigma$.

Now we look at α_2, and write it as $\alpha_2 = \alpha'_2 + \alpha''_2$ where

$$\text{(3)} \qquad \text{(a)} \quad \alpha'_2 \perp \mu \qquad \text{(b)} \quad \alpha''_2 \ll \mu .$$

As before, $\alpha'_2 \ll \alpha_2 \ll \nu$, thus (3) (a), (1) (d) imply $\alpha'_2 \perp \nu_2$, and hence $\alpha'_2 \ll \nu_1$. Also $\alpha'_2 \perp \nu_2$ and $\alpha'_2 \perp \mu$ imply $\alpha'_2 \perp \nu_2 - i\mu$. Therefore by Theorem 2.2 and Lemma 1, there exists a sequence $\{g'_n\}$ of $(S, \Sigma: C)$-simple functions such that

$$\text{(B)} \qquad \lim_n \int_E g'_n \, d(\nu_1 + (\nu_2 - i\mu)) = \lim_n \int_E (-i) g'_n d\beta = \alpha'_2(E)$$

uniformly for $E \in \Sigma$.

We are still left with α''_2. But (3) (b), $\alpha''_2 \ll \alpha_2$, (2) (b), and (1) (a) imply $\alpha''_2 \perp \mu_1$, hence $\alpha''_2 \ll \mu_2$. Hence there exists a sequence $\{k_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E g_n d\mu_2 = \alpha''_2(E)$ uniformly for $E \in \Sigma$. But we cannot apply Lemma 1 here since we do not have $\alpha''_2 \perp \mu_1 + i\nu$. However, (1) (d), $\nu_2 \ll \nu$, and (1) (a) imply $\nu_2 \ll \mu_2$. Therefore there exists a sequence $\{l_n\}$ of $(S, \Sigma: R)$-simple functions such that $\lim_n \int_E l_n d\mu_2 = \nu_2(E)$ uniformly for $E \in \Sigma$. Let $K_n = \max \{| k_n(x) |, x \in X\}$, and, by taking a subsequence of $\{l_n\}$ if necessary, suppose $v\left(\int l_n d\mu_2 - \nu_2, E\right) < 1/nK_n$ for all $E \in \Sigma$. Consider the sequence $\{h_n\}$ where $h_n = k_n/(1 + il_n)$. Notice that h_n is a step function, is defined everwhere since $1 + il_n$ is never zero, and $| h_n(x) | \leqq | k_n(x) |$ for each $x \in S$. Let $\varepsilon > 0$, and choose N so large that $n \geqq N$ implies both

(i) $\left|\alpha''_2(E) - \int_E k_n d\mu_2\right| < \varepsilon/2$ for all $E \in \Sigma$ and

(ii) $1/n < \varepsilon/2$.

If $E \in \Sigma$, then $\left|\alpha''_2(E) - \int_E h_n \, d(\mu_2 + i\nu_2)\right.$

$$\leqq \left|\alpha''_2(E) - \int_E k_n d\mu_2\right| + \left|\int_E k_n d\mu_2 - \int_E h_n(1 + il_n) d\mu_2\right|$$

$$+\left|\int_E h_n(1+il_n)d\mu_2 - \int_E h_n\, d(\mu_2+i\nu_2)\right|$$

$$<\frac{\varepsilon}{2}+0+\left|\int_E ih_nl_nd\mu_2 - \int_E ih_nd\nu_2\right|$$

$$\leqq \frac{\varepsilon}{2}+K_n\cdot\nu\left(\int l_nd\mu_2-\nu_2, E\right) < \frac{\varepsilon}{2}+K_n\cdot\frac{1}{nK_n}$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \text{ if } n\geqq N.$$

Therefore

$$\lim_n\int_E h_nd(\mu_2+i\nu_2)=\alpha_2''(E)$$

uniformly for $E\in\Sigma$.

Noting that $\alpha_2''\perp\mu_1+i\nu_1$ and applying Lemma 1 again yields the existence of a sequence $\{h_n'\}$ of $(S,\Sigma\colon C)$-simple functions such that

$$\text{(C)}\qquad \lim_n\int_E h_n'd(\mu_2+i\nu_2+(\mu_1+i\nu_1))=\lim_n\int_E h_n'd\beta=\alpha_2''(E)$$

uniformly for $E\in\Sigma$.

Considering (A), (B), and (C) together we have

$$\lim_n\int_E (f_n'+(-ig_n')+h_n')d\beta=(\alpha_1+(\alpha_2'+\alpha_2''))(E)=\alpha(E)$$

uniformly for $E\in\Sigma$. Thus, let $z_n=(f_n'+i(-g_n')+h_n')$ and the theorem is proved.

REFERENCES

1. S. Bochner, *Additive set functions on groups*, Ann. of Math. **40** (1939), 769-799.
2. S. Bochner and R. S. Phillips, *Additive set functions and vector lattices*, Ann. of Math. **42** (1941), 316-324.
3. R. B. Darst, *A decomposion of finitely additive set functions*, J. Math. Reine Angew. **210** (1962), 31-37.
4. ———, *A decomposition for complete normed abelian groups with applications to spaces of additive set functions*, Trans. Amer. Math. Soc. **103** (1962), 549-559.
5. ———, *The Lebesgue decomposition*, Duke Math. J. **30** (1963), 553-556.
6. N. Dunford and J. T. Schwartz, *Linear Operators*, Interscience, New York, 1958.
7. C. Fefferman, *A Radon-Nikodym Theorem for finitely additive set functions*, Pacific J. Math. **23** (1967), 35-45.
8. S. Leader, *The theory of L^p-spaces for finitely additive set functions*, Ann. of Math. **58** (1953), 528-543.

Received January 10, 1968. R. B. Darst was partially supported by a NSF grant, and Euline Green was partially supported by a NDEA Title IV fellowship.

PURDUE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH B. H. NEUMANN F. WOLF K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 27, No. 2 February, 1968

Leonard E. Baum and George Roger Sell, *Growth transformations for functions on manifolds* 211
Henry Gilbert Bray, *A note on CLT groups* 229
Paul Robert Chernoff, Richard Anthony Rasala and William Charles Waterhouse, *The Stone-Weierstrass theorem for valuable fields* 233
Douglas Napier Clark, *On matrices associated with generalized interpolation problems* 241
Richard Brian Darst and Euline Irwin Green, *On a Radon-Nikodym theorem for finitely additive set functions* 255
Carl Louis DeVito, *A note on Eberlein's theorem* 261
P. H. Doyle, III and John Gilbert Hocking, *Proving that wild cells exist* 265
Leslie C. Glaser, *Uncountably many almost polyhedral wild $(k-2)$-cells in E^k for $k \geq 4$* 267
Samuel Irving Goldberg, *Totally geodesic hypersurfaces of Kaehler manifolds* 275
Donald Goldsmith, *On the multiplicative properties of arithmetic functions* 283
Jack D. Gray, *Local analytic extensions of the resolvent* 305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, *Commutativity theorems for nonassociative rings with a finite division ring homomorphic image* 325
André (Piotrowsky) De Korvin, *Normal expectations in von Neumann algebras* 333
James Donald Kuelbs, *A linear transformation theorem for analytic Feynman integrals* 339
W. Kuich, *Quasi-block-stochastic matrices* 353
Richard G. Levin, *On commutative, nonpotent archimedean semigroups* 365
James R. McLaughlin, *Functions represented by Rademacher series* 373
Calvin R. Putnam, *Singular integrals and positive kernels* 379
Harold G. Rutherford, II, *Characterizing primes in some noncommutative rings* 387
Benjamin L. Schwartz, *On interchange graphs* 393
Satish Shirali, *On the Jordan structure of complex Banach *algebras* 397
Earl J. Taft, *A counter-example to a fixed point conjecture* 405
J. Roger Teller, *On abelian pseudo lattice ordered groups* 411