

Pacific Journal of Mathematics

A NOTE ON EBERLEIN'S THEOREM

CARL LOUIS DEVITO

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This paper is concerned with locally convex spaces which are closed, separable subspaces of their strong biduals. Let E be a space of this type. We first prove that, for an element of E' , weak* continuity on E is equivalent to sequential weak* continuity on the convex, strongly bounded subsets of E' . We then prove Eberlein's theorem for spaces of this type; i.e., we prove that, for the weakly closed subsets of E , countable weak compactness coincides with weak compactness. Finally, we show that the separability hypothesis in our first theorem is necessary.

Our notation and terminology will be that of [1]. The letter E will always denote a locally convex, topological vector space over the field of real numbers. If we want to call attention to a specific, locally convex topology t on E , we will write $E[t]$. The dual of E will be denoted by E' . The weakest topology on E which renders each element of E' continuous will be denoted by $\sigma(E, E')$. We shall be working with the strong topology, $\beta(E', E)$, on E' . This is the topology of uniform convergence on the convex, $\sigma(E, E')$ -bounded subsets of E . E'' will denote the dual of $E'[\beta(E', E)]$. We shall often identify E with its canonical image in E'' . The topology induced on E by its strong bidual, $E''[\beta(E'', E')]$, will be denoted by $\beta^*(E, E')$. Recall that $\beta^*(E, E')$ is the topology of uniform convergence on the convex, $\beta(E', E)$ -bounded subsets of E' .

DEFINITION. We shall say that E has property (S) if the following is true: An element w of E'' is in E if and only if $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero.

THEOREM 1. *Suppose that $E[\beta^*(E, E')]$ is separable. Then E has property (S) if and only if E is a closed, linear subspace of $E''[\beta(E'', E')]$.*

Proof. We shall prove sufficiency first. Let w be in E'' and suppose that $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero. Let B be a convex, $\beta(E', E)$ -bounded subset of E' and let F be the dual of $E[\beta^*(E, E')]$. Clearly $E' \subset F$ and, by [1; Prop. 2, p. 65], B is relatively $\sigma(F, E)$ -compact. Since E is $\beta^*(E, E')$ -separable, the restriction of $\sigma(F, E)$ to B is metrizable. Hence $\sigma(E', E)$ is metrizable on every

convex, $\beta(E', E)$ -bounded subset of E' . This fact, together with our assumptions on w , implies that w is $\sigma(E', E)$ -continuous on every convex, $\beta(E', E)$ -bounded subset of E' . Thus, by [4; Th. 10, p. 97], w is in the completion of $E[\beta^*(E, E')]$. But w is in E'' and E is closed in $E''[\beta(E'', E')]$. It follows that w is in E .

Now assume that E has property (S). Let w be a point in the closure of E for $E''[\beta(E'', E')]$, and let $\{f_n\}$ be a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero. We may, for each fixed positive integer k , choose x_k in E such that: (a) $|wf_n - x_k f_n| \leq 1/k$ for every n . The inequality

$$|wf_n - wf_m| \leq |wf_n - x_k f_n| + |x_k f_n - x_k f_m| + |x_k f_m - wf_m|$$

shows that $\lim wf_n$ exists. But by (a), this limit is $\leq 1/k$ for every k . Thus, E is closed in $E''[\beta(E'', E')]$.

THEOREM 2. *If E has property (S), then every weakly closed, countably weakly compact subset of E is weakly compact.*

Proof. Let M be a weakly closed, countably weakly compact subset of E . Let w be a point in the closure of M for $E''[\sigma(E'', E')]$ and let $\{f_n\}$ be a sequence of points of E' which is $\beta(E', E)$ -bounded and $\sigma(E', E)$ -convergent to zero. For each positive integer k we may choose x_k in M such that: $|x_k f_n - wf_n| \leq 1/k$ for $n \leq k$. Thus, for each fixed n , $\lim x_k f_n = wf_n$. Since M is countably weakly compact $\{x_k\}$ has a weak adherent point x_0 in M . It follows that $wf_n = x_0 f_n$ for every n . But then $\lim wf_n = 0$ and, since E has property (S), w is in E and hence in M .

Let B be a Banach space and let Q be a linear subspace of B' . Following Dixmier [2], we shall say that Q has positive characteristic if $\{x \text{ in } Q \mid \|x\| \leq 1\}$ is weak* dense in some ball of B' . If Q has positive characteristic and is also norm closed in B' , then it is easily seen that $\beta^*(B, Q)$ is equivalent to the norm topology of B . Thus, if B is separable, then Theorem 2 shows that compactness and countable compactness coincide for the closed subsets of $B[\sigma(B, Q)]$. This result was first obtained by I. Singer [6] who also showed that it is no longer true if B is nonseparable; see [7]. Hence, in Theorem 1, the separability of $E[\beta^*(E, E')]$ is necessary.

In the preceding application we made use of the following:

THEOREM 3. *If $E[\beta^*(E, E')]$ is both complete and separable, then E has property (S).*

Y. Komura [5] has shown that the strong bidual of a locally convex space need not be complete. Thus Theorem 3 is weaker than Theorem 1.

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Pacific Journal of Mathematics

Vol. 27, No. 2

February, 1968

Leonard E. Baum and George Roger Sell, <i>Growth transformations for functions on manifolds</i>	211
Henry Gilbert Bray, <i>A note on CLT groups</i>	229
Paul Robert Chernoff, Richard Anthony Rasala and William Charles Waterhouse, <i>The Stone-Weierstrass theorem for valuable fields</i>	233
Douglas Napier Clark, <i>On matrices associated with generalized interpolation problems</i>	241
Richard Brian Darst and Euline Irwin Green, <i>On a Radon-Nikodym theorem for finitely additive set functions</i>	255
Carl Louis DeVito, <i>A note on Eberlein's theorem</i>	261
P. H. Doyle, III and John Gilbert Hocking, <i>Proving that wild cells exist</i>	265
Leslie C. Glaser, <i>Uncountably many almost polyhedral wild $(k - 2)$-cells in E^k for $k \geq 4$</i>	267
Samuel Irving Goldberg, <i>Totally geodesic hypersurfaces of Kaehler manifolds</i>	275
Donald Goldsmith, <i>On the multiplicative properties of arithmetic functions</i>	283
Jack D. Gray, <i>Local analytic extensions of the resolvent</i>	305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, <i>Commutativity theorems for nonassociative rings with a finite division ring homomorphic image</i>	325
André (Piotrowsky) De Korvin, <i>Normal expectations in von Neumann algebras</i>	333
James Donald Kuelbs, <i>A linear transformation theorem for analytic Feynman integrals</i>	339
W. Kuich, <i>Quasi-block-stochastic matrices</i>	353
Richard G. Levin, <i>On commutative, nonpotent archimedean semigroups</i>	365
James R. McLaughlin, <i>Functions represented by Rademacher series</i>	373
Calvin R. Putnam, <i>Singular integrals and positive kernels</i>	379
Harold G. Rutherford, II, <i>Characterizing primes in some noncommutative rings</i>	387
Benjamin L. Schwartz, <i>On interchange graphs</i>	393
Satish Shirali, <i>On the Jordan structure of complex Banach* algebras</i>	397
Earl J. Taft, <i>A counter-example to a fixed point conjecture</i>	405
J. Roger Teller, <i>On abelian pseudo lattice ordered groups</i>	411