

Pacific Journal of Mathematics

A NOTE ON EBERLEIN'S THEOREM

CARL LOUIS DEVITO

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This paper is concerned with locally convex spaces which are closed, separable subspaces of their strong biduals. Let E be a space of this type. We first prove that, for an element of E'' , weak* continuity on E' is equivalent to sequential weak* continuity on the convex, strongly bounded subsets of E' . We then prove Eberlein's theorem for spaces of this type; i.e., we prove that, for the weakly closed subsets of E , countable weak compactness coincides with weak compactness. Finally, we show that the separability hypothesis in our first theorem is necessary.

Our notation and terminology will be that of [1]. The letter E will always denote a locally convex, topological vector space over the field of real numbers. If we want to call attention to a specific, locally convex topology t on E , we will write $E[t]$. The dual of E will be denoted by E' . The weakest topology on E which renders each element of E' continuous will be denoted by $\sigma(E, E')$. We shall be working with the strong topology, $\beta(E', E)$, on E' . This is the topology of uniform convergence on the convex, $\sigma(E, E')$ -bounded subsets of E . E'' will denote the dual of $E'[\beta(E', E)]$. We shall often identify E with its canonical image in E'' . The topology induced on E by its strong bidual, $E''[\beta(E'', E')]$, will be denoted by $\beta^*(E, E')$. Recall that $\beta^*(E, E')$ is the topology of uniform convergence on the convex, $\beta(E', E)$ -bounded subsets of E' .

DEFINITION. We shall say that E has property (S) if the following is true: An element w of E'' is in E if and only if $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero.

THEOREM 1. *Suppose that $E[\beta^*(E, E')]$ is separable. Then E has property (S) if and only if E is a closed, linear subspace of $E''[\beta(E'', E')]$.*

Proof. We shall prove sufficiency first. Let w be in E'' and suppose that $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero. Let B be a convex, $\beta(E', E)$ -bounded subset of E' and let F be the dual of $E[\beta^*(E, E')]$. Clearly $E' \subset F$ and, by [1; Prop. 2, p. 65], B is relatively $\sigma(F, E)$ -compact. Since E is $\beta^*(E, E')$ -separable, the restriction of $\sigma(F, E)$ to B is metrizable. Hence $\sigma(E', E)$ is metrizable on every

convex, $\beta(E' E)$ -bounded subset of E' . This fact, together with our assumptions on w , implies that w is $\sigma(E', E)$ -continuous on every convex, $\beta(E', E)$ -bounded subset of E' . Thus, by [4; Th. 10, p. 97], w is in the completion of $E[\beta^*(E, E')]$. But w is in E'' and E is closed in $E''[\beta(E'', E')]$. It follows that w is in E .

Now assume that E has property (S). Let w be a point in the closure of E for $E''[\beta(E'', E')]$, and let $\{f_n\}$ be a $\beta(E', E)$ -bounded sequence of points of E' which is $\sigma(E', E)$ -convergent to zero. We may, for each fixed positive integer k , choose x_k in E such that: (a) $|wf_n - x_k f_n| \leq 1/k$ for every n . The inequality

$$|wf_n - wf_m| \leq |wf_n - x_k f_n| + |x_k f_n - x_k f_m| + |x_k f_m - wf_m|$$

shows that $\lim wf_n$ exists. But by (a), this limit is $\leq 1/k$ for every k . Thus, E is closed in $E''[\beta(E'', E')]$.

THEOREM 2. *If E has property (S), then every weakly closed, countably weakly compact subset of E is weakly compact.*

Proof. Let M be a weakly closed, countably weakly compact subset of E . Let w be a point in the closure of M for $E''[\sigma(E'', E')]$ and let $\{f_n\}$ be a sequence of points of E' which is $\beta(E', E)$ -bounded and $\sigma(E', E)$ -convergent to zero. For each positive integer k we may choose x_k in M such that: $|x_k f_n - wf_n| \leq 1/k$ for $n \leq k$. Thus, for each fixed n , $\lim x_k f_n = wf_n$. Since M is countably weakly compact $\{x_k\}$ has a weak adherent point x_0 in M . It follows that $wf_n = x_0 f_n$ for every n . But then $\lim wf_n = 0$ and, since E has property (S), w is in E and hence in M .

Let B be a Banach space and let Q be a linear subspace of B' . Following Dixmier [2], we shall say that Q has positive characteristic if $\{x \text{ in } Q \mid \|x\| \leq 1\}$ is weak* dense in some ball of B' . If Q has positive characteristic and is also norm closed in B' , then it is easily seen that $\beta^*(B, Q)$ is equivalent to the norm topology of B . Thus, if B is separable, then Theorem 2 shows that compactness and countable compactness coincide for the closed subsets of $B[\sigma(B, Q)]$. This result was first obtained by I. Singer [6] who also showed that it is no longer true if B is nonseparable; see [7]. Hence, in Theorem 1, the separability of $E[\beta^*(E, E')]$ is necessary.

In the preceding application we made use of the following:

THEOREM 3. *If $E[\beta^*(E, E')]$ is both complete and separable, then E has property (S).*

Y. Komura [5] has shown that the strong bidual of a locally convex space need not be complete. Thus Theorem 3 is weaker than Theorem 1.

REFERENCES

1. N. Bourbaki, *Espaces vectoriels topologiques*, Act. Sci. et Indust. No. 1229.
2. J. Dixmier, *Sur un theoreme de Banach*, Duke Math. **15** (1948), 1057-1071.
3. W. F. Eberlein, *Weak compactness in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 51-53.
4. A. Grothendieck, *Espaces vectoriels topologiques*. Soc. Math. S. Paulo, 1958.
5. Y. Komura, *Some examples on linear topological spaces*, Math. Annalen **153** (1964), 150-162.
6. I. Singer, *Weak compactness, pseudo-reflexivity, and quasi-reflexivity*, Math. Annalen **154** (1964) 77-87.
7. *On Banach spaces reflexive with respect to a linear subspace of their conjugate space*. Bull. Math. Soc. Sci. Math. Phys. R.P.S. **2** (1958), 449-462.

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