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**This paper is concerned with locally convex spaces which are closed, separable subspaces of their strong biduals. Let $E$ be a space of this type. We first prove that, for an element of $E''$, weak* continuity on $E'$ is equivalent to sequential weak* continuity on the convex, strongly bounded subsets of $E'$. We then prove Eberlein's theorem for spaces of this type; i.e., we prove that, for the weakly closed subsets of $E$, countable weak compactness coincides with weak compactness. Finally, we show that the separability hypothesis in our first theorem is necessary.**

Our notation and terminology will be that of [1]. The letter $E$ will always denote a locally convex, topological vector space over the field of real numbers. If we want to call attention to a specific, locally convex topology $t$ on $E$, we will write $E[t]$. The dual of $E$ will be denoted by $E'$. The weakest topology on $E$ which renders each element of $E'$ continuous will be denoted by $\sigma(E, E')$. We shall be working with the strong topology, $\beta(E', E)$, on $E'$. This is the topology of uniform convergence on the convex, $\sigma(E, E')$-bounded subsets of $E$. $E''$ will denote the dual of $E'[\beta(E', E)]$. We shall often identify $E$ with its canonical image in $E''$. The topology induced on $E$ by its strong bidual, $E''[\beta(E'', E')]$, will be denoted by $\beta^*(E, E')$. Recall that $\beta^*(E, E')$ is the topology of uniform convergence on the convex, $\beta(E', E)$-bounded subsets of $E'$.

DEFINITION. We shall say that $E$ has property $(S)$ if the following is true: An element $w$ of $E''$ is in $E$ if and only if $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of $E'$ which is $\sigma(E', E)$-convergent to zero.

THEOREM 1. *Suppose that $E[\beta^*(E, E')]$ is separable. Then $E$ has property $(S)$ if and only if $E$ is a closed, linear subspace of $E''[\beta(E'', E')]$.*

*Proof.* We shall prove sufficiency first. Let $w$ be in $E''$ and suppose that $\lim wf_n = 0$, whenever $\{f_n\}$ is a $\beta(E', E)$-bounded sequence of points of $E'$ which is $\sigma(E', E)$-convergent to zero. Let $B$ be a convex, $\beta(E', E)$-bounded subset of $E'$ and let $F$ be the dual of $E[\beta^*(E, E')]$. Clearly $E' \subset F$ and, by [1; Prop. 2, p. 65], $B$ is relatively $\sigma(F, E)$-compact. Since $E$ is $\beta^*(E, E')$-separable, the restriction of $\sigma(F, E)$ to $B$ is metrizable. Hence $\sigma(E', E)$ is metrizable on every convex, $\beta(E' E)$-bounded subset of $E'$. This fact, together with our assumptions on $w$, implies that $w$ is $\sigma(E', E)$-continuous on every convex, $\beta(E', E)$-bounded subset of $E'$. Thus, by [4; Th. 10, p. 97], $w$ is in the completion of $E[\beta^*(E, E')]$. But $w$ is in $E''$ and $E$ is closed in $E''[\beta(E'', E')]$. It follows that $w$ is in $E$.

Now assume that $E$ has property $(S)$. Let $w$ be a point in the closure of $E$ for $E'[\beta(E'', E')]$, and let $\{f_n\}$ be a $\beta(E', E)$-bounded sequence of points of $E'$ which is $\sigma(E', E)$-convergent to zero. We may, for each fixed positive integer $k$, choose $x_k$ in $E$ such that: (a) $| wf_n - x_k f_n | \leqq 1/k$ for every $n$. The inequality

$$| wf_n - wf_m | \leqq | wf_n - x_k f_n | + | x_k f_n - x_k f_m | + | x_k f_m - wf_m |$$

shows that $\lim wf_n$ exists. But by (a), this limit is $\leqq 1/k$ for every $k$. Thus, $E$ is closed in $E''[\beta(E'', E')]$.

THEOREM 2. *If $E$ has property $(S)$, then every weakly closed, countably weakly compact subset of $E$ is weakly compact.*

*Proof.* Let $M$ be a weakly closed, countably weakly compact subset of $E$. Let $w$ be a point in the closure of $M$ for $E''[\sigma(E'', E')]$ and let $\{f_n\}$ be a sequence of points of $E'$ which is $\beta(E', E)$-bounded and $\sigma(E', E)$-convergent to zero. For each positive integer $k$ we may choose $x_k$ in $M$ such that: $| x_k f_n - wf_n | \leqq 1/k$ for $n \leqq k$. Thus, for each fixed $n$, $\lim x_k f_n = wf_n$. Since $M$ is countably weakly compact $\{x_k\}$ has a weak adherent point $x_0$ in $M$. It follows that $wf_n = x_0 f_n$ for every $n$. But then $\lim wf_n = 0$ and, since $E$ has property $(S)$, $w$ is in $E$ and hence in $M$.

Let $B$ be a Banach space and let $Q$ be a linear subspace of $B'$. Following Dixmier [2], we shall say that $Q$ has positive characteristic if $\{x \text{ in } Q \mid \|x\| \leqq 1\}$ is weak* dense in some ball of $B'$. If $Q$ has positive characteristic and is also norm closed in $B'$, then it is easily seen that $\beta^*(B, Q)$ is equivalent to the norm topology of $B$. Thus, if $B$ is separable, then Theorem 2 shows that compactness and countable compactness coincide for the closed subsets of $B[\sigma(B, Q)]$. This result was first obtained by I. Singer [6] who also showed that it is no longer true if $B$ is nonseparable; see [7]. Hence, in Theorem 1, the separability of $E[\beta^*(E, E')]$ is necessary.

In the preceding application we made use of the following:

THEOREM 3. *If $E[\beta^*(E, E')]$ is both complete and separable, then $E$ has property $(S)$.*

Y. Komura [5] has shown that the strong bidual of a locally convex space need not be complete. Thus Theorem 3 is weaker than Theorem 1.

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