PROVING THAT WILD CELLS EXIST

P. H. Doyle, III and John Gilbert Hocking
PROVING THAT WILD CELLS EXIST

P. H. DOYLE AND J. G. HOCKING

In their famous paper Fox and Artin constructed several examples of wild cells in 3-space. The present authors construct a wild disk \( D \) in the 4-sphere \( S^4 \) with the property that the proof of nontameness is perhaps the most elementary possible. We require only the knowledge that if \( K \) is the trefoil knot in the 3-sphere \( S^3 \), then the fundamental group \( \pi_1(S^3 - K) \) is not abelian. Parenthetically, the wild disk \( D \) constructed here has the property that every arc on \( D \) is tame, a fact which follows immediately from the construction.

In \( S^3 \) let \( \{K_i\} \) be a sequence of polygonal trefoil knots that converge to a point \( q \) while each \( K_i \) lies interior to a 3-simplex that meets no other \( K_i \). We consider \( S^3 \) as being the equator of \( S^4 \) while \( H \) is the upper hemisphere of \( S^4 \). In \( H - S^3 \) let \( \{p_i\} \) be a sequence of points converging to \( q \). If \( p_iK_i \) is the cone over \( K_i \) with vertex \( p_i \), let \( \{p_i\} \) be so chosen that the disks \( \{p_iK_i\} \) are disjoint in pairs. Now in \( S^3 \) join \( p_iK_i \) and \( p_iK_2 \) by a polyhedral disk \( D_1 \) so that \( p_iK_1 \cup D_1 \cup p_iK_2 \) is a disk disjoint from \( (\bigcup_i p_iK_i) \cup q \). We next join \( p_iK_2 \) and \( p_iK_3 \) by a polyhedral disk, \( D_2 \), in \( S^3 \) so that \( p_iK_1 \cup D_1 \cup p_iK_2 \cup D_2 \cup p_iK_3 \) is a disk disjoint from \( (\bigcup_i p_iK_i) \cup q \). This process is continued so that as \( i \to \infty \) the diameter of \( D_i \) tends to 0 and the disk \( D \) is

\[
\left( \bigcup_i (p_iK_i \cup D_i) \right) \cup q.
\]

As a subset of \( S^4 \), \( D \) is locally tame [1] except perhaps at \( q \).

**Theorem.** \( D \) is wild in \( S^4 \).

The proof is given in two lemmas.

**Lemma 1.** If there is a homeomorphism \( h \) of \( S^4 \) onto \( S^4 \) such that \( h(D) \) is the union of a finite number of triangles, then for some point \( p \) in \( D \) there is a neighborhood \( U \) of \( p \) in \( D \) and for each open set \( V \) containing \( p \) there is a neighborhood \( V_1 \) of \( p \) such that \( \pi_1(V_1 - U) \) is abelian.

**Proof.** If \( h \) exists then \( \{h(p_i)\} \) contains a point that lies in the interior of a disk formed by the union of two triangles. Call this point \( h(p_i) \). Then \( p_i \) has a neighborhood meeting the condition in the lemma while \( \pi_1(V_1 - U) \) is the infinite cyclic group.

265
LEMMA 2. No point $p_3$ in the disk $D$ meets the conclusion of Lemma 1.

Proof. If such a point were to exist we note that if $K$ is a polygonal trefoil in $S^3$, $S^4$ the suspension of $S^3$, then the suspension of $K$ in $S^4$ is a 2-sphere $S^2$. But $\pi_1(S^3 - K) \cong \pi_1(S^4 - S^2)$. All generators of $\pi_1(S^4 - S^2)$ may be selected in $S^3 - K$. So if Lemma 1 holds at a suspension vertex of $K$, $\pi_1(S^4 - S^2)$ is abelian.

Essentially the same construction yields a wild $(n - 2)$-disk in $S^n$ for $n \geq 4$.

REFERENCES

1. R. H. Bing, Locally tame sets are tame, Ann. of Math. 59 (1954), 145-158.

Received January 4, 1968. Work on this note was supported by NSF Grant GP-7126.

Michigan State University
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsuisha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.
Leonard E. Baum and George Roger Sell, *Growth transformations for functions on manifolds* .................................................. 211
Henry Gilbert Bray, *A note on CLT groups* ................................................. 229
Douglas Napier Clark, *On matrices associated with generalized interpolation problems* .................................................... 241
Richard Brian Darst and Euline Irwin Green, *On a Radon-Nikodym theorem for finitely additive set functions* .................................................. 255
Carl Louis DeVito, *A note on Eberlein’s theorem* ........................................ 261
P. H. Doyle, III and John Gilbert Hocking, *Proving that wild cells exist* .......... 265
Leslie C. Glaser, *Uncountably many almost polyhedral wild (k − 2)-cells in E^k for k ≥ 4* .................................................. 267
Samuel Irving Goldberg, *Totally geodesic hypersurfaces of Kaehler manifolds* ................................................................. 275
Donald Goldsmith, *On the multiplicative properties of arithmetic functions* ................................................................. 283
Jack D. Gray, *Local analytic extensions of the resolvent* ......................... 305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, *Commutativity theorems for nonassociative rings with a finite division ring homomorphic image* .................................................. 325
André (Piotrowsky) De Korvin, *Normal expectations in von Neumann algebras* ........................................................................... 333
James Donald Kuelbs, *A linear transformation theorem for analytic Feynman integrals* ........................................................................ 339
W. Kuich, *Quasi-block-stochastic matrices* .............................................. 353
Richard G. Levin, *On commutative, nonpotent archimedean semigroups* ................................................................................. 365
James R. McLaughlin, *Functions represented by Rademacher series* ........ 373
Calvin R. Putnam, *Singular integrals and positive kernels* .......................... 379
Harold G. Rutherford, II, *Characterizing primes in some noncommutative rings* .......................................................................... 387
Benjamin L. Schwartz, *On interchange graphs* ......................................... 393
Satish Shirali, *On the Jordan structure of complex Banach*^*^algebras* ........ 397
Earl J. Taft, *A counter-example to a fixed point conjecture* .......................... 405
J. Roger Teller, *On abelian pseudo lattice ordered groups* ........................... 411