

# Pacific Journal of Mathematics

**PROVING THAT WILD CELLS EXIST**

P. H. DOYLE, III AND JOHN GILBERT HOCKING

## PROVING THAT WILD CELLS EXIST

P. H. DOYLE AND J. G. HOCKING

**In their famous paper Fox and Artin constructed several examples of wild cells in 3-space. The present authors construct a wild disk  $D$  in the 4-sphere  $S^4$  with the property that the proof of nontameness is perhaps the most elementary possible. We require only the knowledge that if  $K$  is the trefoil knot in the 3-sphere  $S^3$ , then the fundamental group  $\pi_1(S^3 - K)$  is not abelian. Parenthetically, the wild disk  $D$  constructed here has the property that every arc on  $D$  is tame, a fact which follows immediately from the construction.**

In  $S^3$  let  $\{K_i\}$  be a sequence of polygonal trefoil knots that converge to a point  $q$  while each  $K_j$  lies interior to a 3-simplex that meets no other  $K_i$ . We consider  $S^3$  as being the equator of  $S^4$  while  $H$  is the upper hemisphere of  $S^4$ . In  $H - S^3$  let  $\{p_i\}$  be a sequence of points converging to  $q$ . If  $p_i K_i$  is the cone over  $K_i$  with vertex  $p_i$ , let  $\{p_i\}$  be so chosen that the disks  $\{p_i K_i\}$  are disjoint in pairs. Now in  $S^3$  join  $p_1 K_1$  and  $p_2 K_2$  by a polyhedral disk  $D_1$  so that  $p_1 K_1 \cup D_1 \cup p_2 K_2$  is a disk disjoint from  $(\bigcup_3^\infty p_i K_i) \cup q$ . We next join  $p_2 K_2$  and  $p_3 K_3$  by a polyhedral disk,  $D_2$ , in  $S^3$  so that  $p_1 K_1 \cup D_1 \cup p_2 K_2 \cup D_2 \cup p_3 K_3$  is a disk disjoint from  $(\bigcup_i^\infty p_i K_i) \cup q$ . This process is continued so that as  $i \rightarrow \infty$  the diameter of  $D_i$  tends to 0 and the disk  $D$  is

$$\left( \bigcup_1^\infty (p_i K_i \cup D_i) \right) \cup q .$$

As a subset of  $S^4$ ,  $D$  is locally tame [1] except perhaps at  $q$ .

**THEOREM.**  *$D$  is wild in  $S^4$ .*

The proof is given in two lemmas.

**LEMMA 1.** *If there is a homeomorphism  $h$  of  $S^4$  onto  $S^4$  such that  $h(D)$  is the union of a finite number of triangles, then for some point  $p_j$  in  $D$  there is a neighborhood  $U_j$  of  $p_j$  in  $D$  and for each open set  $V'_j$  in  $S^4$  containing  $p_j$  there is a neighborhood  $V_j \subset V'_j$  of  $p_j$  such that  $\pi_1(V_j - U_j)$  is abelian.*

*Proof.* If  $h$  exists then  $\{h(p_i)\}$  contains a point that lies in the interior of a disk formed by the union of two triangles. Call this point  $h(p_j)$ . Then  $p_j$  has a neighborhood meeting the condition in the lemma while  $\pi_1(V_j - U_j)$  is the infinite cyclic group.

LEMMA 2. *No point  $p_j$  in the disk  $D$  meets the conclusion of Lemma 1.*

*Proof.* If such a point were to exist we note that if  $K$  is a polygonal trefoil in  $S^3$ ,  $S^4$  the suspension of  $S^3$ , then the suspension of  $K$  in  $S^4$  is a 2-sphere  $S^2$ . But  $\pi_1(S^3 - K) \cong \pi_1(S^4 - S^2)$ . All generators of  $\pi_1(S^4 - S^2)$  may be selected in  $S^3 - K$ . So if Lemma 1 holds at a suspension vertex of  $K$ ,  $\pi_1(S^4 - S^2)$  is abelian.

Essentially the same construction yields a wild  $(n - 2)$ -disk in  $S^n$  for  $n \geq 4$ .

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