

Pacific Journal of Mathematics

NORMAL EXPECTATIONS IN VON NEUMANN ALGEBRAS

ANDRÉ (PIOTROWSKY) DE KORVIN

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Let h and k be two Hilbert spaces, $h \otimes k$ will denote the tensor product of h and k . Let \mathcal{A} be a von Neumann algebra acting on h . Let ψ be an ampliation of \mathcal{A} in $h \otimes k$, i.e., ψ is a map of \mathcal{A} into bounded linear operators of $h \otimes k$ and $\psi(\mathcal{A}) = \mathcal{A} \otimes I_k$ (I_k is the identity map on k). Let $\tilde{\mathcal{A}}$ be the image of \mathcal{A} by ψ .

The purpose of this paper is to prove the following result: If \mathcal{B} is a subalgebra of \mathcal{A} and if \mathcal{B} is the range of a normal expectation φ defined on \mathcal{A} , then there exists an ampliation of \mathcal{A} in $h \otimes k$, independent of \mathcal{B} and of φ , such that $\varphi \otimes I_k$ is a spatial isomorphism of $\tilde{\mathcal{A}}$.

Let \mathcal{A} and \mathcal{B} be two C^* algebras with identity. Suppose $\mathcal{B} \subset \mathcal{A}$. Let φ be a positive linear map of \mathcal{A} on \mathcal{B} such that φ preserves the identity and such that $\varphi(BX) = B\varphi(X)$ for all B in \mathcal{B} and all X in \mathcal{A} . φ is then defined to be an expectation of \mathcal{A} on \mathcal{B} . The extension of the notion of an expectation in the probability theory sense, to expectations on finite von Neumann algebra is largely due to J. Dixmier and H. Umegaki [1]. In [4] Tomiyama considers an expectation on von Neumann algebras to be a projection of norm one. If φ is an expectation in the sense $\varphi(BX) = B\varphi(X)$, φ positive and φ preserves identities, then $\varphi(XB) = \varphi(X)B$ for all X in \mathcal{A} , B in \mathcal{B} . \mathcal{B} is the set of fixed points of φ . By writing $\varphi[(X - \varphi(X))^*(X - \varphi(X))] \geq 0$ we have $\varphi(X^*X) \geq \varphi(X)^*\varphi(X)$. In particular φ is a bounded map. The result stated in the previous paragraph extends a result by Nakamura, Takesaki, and Umegaki [2], who consider the case when \mathcal{A} is a finite von Neumann algebra.

2. Preliminaries. Basic definitions and some essentially known results will now be given for ready reference. Let M and N be C^* algebras and φ a positive linear map of M on N . Let M_n be the set of all $n \times n$ matrices whose entries are elements of M , call those entries $A_{i,j}$. Define for each n , $\varphi^{(n)}(A_{i,j}) = (\varphi(A_{i,j}))$; φ^n is then a map of M_n on N_n . φ is called *completely positive* if each φ^n is.

Let \mathcal{A} and \mathcal{B} be two von Neumann algebras, with $\mathcal{B} \subset \mathcal{A}$. Let φ be an expectation of \mathcal{A} on \mathcal{B} . φ is called *faithful* if for any T in \mathcal{A} , $\varphi(TT^*) = 0$ implies $T = 0$. Let A_α be a net of uniformly bounded self adjoint operators in \mathcal{A} . φ is called *normal* if

$$\sup_{\alpha} \varphi(A_{\alpha}) = \varphi(\sup_{\alpha} A_{\alpha}).$$

The *ultra-weak topology* on α will be the weakest which will make all $\sum w_{x_i, y_i}(A) = \sum(Ax_i, y_i)$ continuous where

$$\sum \|x_i\|^2 < \infty \quad \text{and} \quad \sum \|y_i\|^2 < \infty .$$

In what follows if N is arbitrary von Neumann algebra, N' will denote the commutant of N . If h is any Hilbert space, $\dim h$ will denote the cardinality of the dimension of h .

LEMMA 1. *Let M and N be two von Neumann algebras acting on h_M and h_N . Let φ be a* isomorphism of M on N . Let k be a Hilbert space such that $\dim k \geq \text{Max}(\chi_1, \dim h_M, \dim h_N)$, then $\varphi \otimes I_k$ is a spatial isomorphism. This theorem says that there exists an isometry V of $h_M \otimes k$ on $h_N \otimes k$ such that*

$$\varphi \otimes I_k(A \otimes I_k) = \varphi(A) \otimes I_k = V(A \otimes I_k)V^*(= V\tilde{A}V^*) .$$

Tomiyama has shown this result in [5].

LEMMA 2. *Let M and N be two C^* algebras with identities. Let φ be an expectation of M on N , then φ is completely positive. This result was shown by Nakamura, Takesaki, and Umegaki in [2].*

One of the tools for the proof of the theorem will be the Stinespring construction which is given in [3] and which will be sketched here for completeness sake.

Let M be any von Neumann algebra acting on h . Let $M \odot h$ denote the tensor product of M and h as linear spaces. Let N be von Neumann algebra of M which is the range of a normal expectation φ . On $M \odot h$ define an inner product by:

$$\langle \sum_{i=1}^n a_i \otimes x_i, \sum_{j=1}^l b_j \otimes y_j \rangle = \sum_{i,j} (\varphi(b_j^* \cdot a_i)x_i, y_j)$$

where a_i, b_j are in M, x_i, y_j are in h and where $(,)$ denotes the inner product in h . Now:

$$\sum_{i,j} (a_j^* a_i x_i, x_j) = (\sum_{i=1}^n a_i x_i, \sum_{i=1}^n a_i x_i) \geq 0 .$$

Let A be in M_n with $A_{ij} = a_j^* a_i$ then if $x = (x_1, x_2, \dots, x_n)$

$$(Ax, x) = \sum_{i,j} (a_j^* a_i x_i, x_j) \geq 0 .$$

By Proposition 2,

$$\sum_{i,j} (\varphi(a_j^* a_i)x_i, x_j) \geq 0 .$$

Hence the inner product defined on $M \odot h$ is bilinear and positive. However, it is possible to have $\langle \zeta, \zeta \rangle = 0$ with $\zeta \neq 0$. Divide out the space $M \odot h$ by all vectors of norm zero. Then taking the completion of that space, one obtains a Hilbert space which will be denoted $M \otimes h$.

LEMMA 3. h is embedded as a Hilbert space in $M \otimes h$.

Proof. In fact we shall show that h is isomorphic to $N \otimes h$. Let $a_i, i = 1, 2, \dots, n$ be operators in N , consider the map

$$S(\sum_{i=1}^n a_i \otimes x_i) = \sum_{i=1}^n a_i x_i$$

then

$$\begin{aligned} & \langle \sum_{i=1}^n a_i \otimes x_i, \sum_{i=1}^n a_i \otimes x_i \rangle \\ &= \sum_{i,j} (\varphi(a_j^* a_i) x_i, x_j) \\ &= \sum_{i,j} (a_j^* a_i x_i, x_j) \\ &= (\sum_{i=1}^n a_i x_i, \sum_{i=1}^n a_i x_i). \end{aligned}$$

Hence S is an isometry of $N \otimes h$ on h . In particular then, one can view h as a subspace of $M \otimes h$.

LEMMA 4. φ defines a self adjoint projection E of $M \otimes h$ on $N \otimes h$.

Proof. Let $a_i, i = 1, 2, \dots, n$ be operators of M . Define

$$E(\sum_{i=1}^n a_i \otimes x_i) = \sum_{i=1}^n \varphi(a_i) \otimes x_i$$

the proof in [2] shows that E is a well-defined self adjoint projection of $M \otimes h$ on $N \otimes h$. Recall for example how self adjointness is checked out.

$$\begin{aligned} & \langle E(\sum_i a_i \otimes x_i), \sum_j b_j \otimes y_j \rangle \\ &= \langle \sum_i \varphi(a_i) \otimes x_i, \sum_j b_j \otimes y_j \rangle = \sum_{i,j} (\varphi(b_j^* \varphi(a_i)) x_i, y_j) \\ &= \sum_{i,j} (\varphi(\varphi(b_j^*) a_i) x_i, y_j) \\ &= \langle \sum a_i \otimes x_i, \sum_j \varphi(b_j) \otimes y_j \rangle \\ &= \langle \sum_i a_i \otimes x_i, E(\sum_j b_j \otimes y_j) \rangle. \end{aligned}$$

LEMMA 5. *There exists an ultra-weakly continuous representation l of M in $L(M \otimes h)$ such that $l(b)E = El(b)$ for all b in N . Moreover if h and $N \otimes h$ are identified by the isometry S of Lemma 3, then $\varphi(A) = El(a)E$ for all a in M .*

Proof. For each a in M define

$$l(a)(\sum a_i \otimes x_i) = \sum a a_i \otimes x_i$$

l is then a representation of M in $L(M \otimes h)$. Let $b_i, i = 1, 2, \dots, n$ be operators in N then:

$$\begin{aligned} El(a)(E b_j \otimes x_j) &= E(\sum a b_j \otimes x_j) \\ &= \sum \varphi(a) b_j \otimes x_j = \varphi(a)(\sum b_j \otimes x_j) \end{aligned}$$

identifying $\sum b_j \otimes x_j$ with $\sum b_j x_j$ this shows that $El(a)E = \varphi(a)$. Let b be in N then

$$\begin{aligned} l(b)E(\sum a_i \otimes x_i) &= l(b)(\sum \varphi(a_i) \otimes x_i) \\ &= \sum b \varphi(a_i) \otimes x_i = El(b)(\sum a_i \otimes x_i) . \end{aligned}$$

So $l(b)E = El(b)$ for all b in N . To show now that l is u. w. continuous, let

$$\zeta_k = \sum_{i=1}^{n_k} a_i^{(k)} \otimes x_i^{(k)}, \eta_h = \sum_{j=1}^{n_h} b_j^{(h)} \otimes y_j^{(h)}$$

with $\sum \|\zeta_k\|^2 < \infty$ and $\sum \|\eta_h\|^2 < \infty$. Let a_α be a net converging u. w. to a in M . Then it is sufficient to show that A tends to zero where

$$A = \sum_{k,h} \langle l(a - a_\alpha) \zeta_k, \eta_h \rangle .$$

we have

$$A = \sum_{k,h} \sum_{i,j} (\varphi(b_j^{(h)*} (a - a_\alpha) a_i^{(k)}) x_i^{(k)}, y_j^{(h)}) .$$

Now $b_j^{(h)*} (a - a_\alpha) a_i^{(k)}$ tends to zero u.w. As φ is normal, A tends to zero. Let $N \subset M$ be two von Neumann algebras acting on h . Let φ be a faithful, normal expectation of M on N .

3. Main results. First the following result will be established.

PROPOSITION 6. There exists a Hilbert space k such that:

- (1) h can be embedded in k .
- (2) There exists an u.w. continuous representation l of M in $L(k)$ such that $\varphi(A) = p_{hl}(A)p_h$ where p_h is the projection of k on h .
- (3) l is a $*$ isomorphism.

(4) p_h commutes with all $l(b)$ with b in N .

Proof. Let $k = M \otimes h$, if $l(a) = 0$ then $l(a^*a) = 0$ so $\varphi(a^*a) = 0$.

By faithfulness of φ , this implies $a = 0$. Hence l is a $*$ isomorphism of M in $L(k)$. The rest of Proposition 6 is a restatement of Lemma 5. The main result of this paper can now be given.

THEOREM 7. *There exists an ampliation of M in $h \otimes k$ such that if N is any von Neumann subalgebra of M which is the range of a normal expectation φ , then there exists an isometry V in $(N \otimes I_k)'$ such that $\varphi \otimes I_k(\tilde{A}) = V\tilde{A}V^*$, $VV^* = I$, on putting $V^*V = P$, then P is in $(N \otimes I_k)'$, $\varphi \otimes I_k(\tilde{A})P = P\tilde{A}P$. If φ is faithful then $\tilde{A}P = 0$ ($A \geq 0$) implies $\tilde{A} = 0$.*

Proof. Let s be a Hilbert space with cardinality greater or equal to the maximum of ψ_1 and cardinality of a Hammet basis of $M \otimes h$. Define $\tilde{l}(\tilde{A}) = l(A) \otimes I_s$, $\tilde{\varphi} = \varphi \otimes I_s$. Then $\tilde{\varphi}(\tilde{A}) = (P_h \otimes I_s)\tilde{l}(\tilde{A})(P_h \otimes I_s)$. By Lemma 1, l is spatial. There exists an isometry U of $h \otimes s$ onto $k \otimes s$ such that $\tilde{\varphi}(\tilde{A}) = U(\tilde{A})U^*$. Hence

$$\tilde{\varphi}(\tilde{A}) = P_{h \otimes s}U(A \otimes I_s)U^*P_{h \otimes s}$$

where $P_{h \otimes s}$ denotes the projection of $k \otimes s$ on $h \otimes s$. Moreover $P_{h \otimes s}$ commutes with all $U\tilde{B}U^*$ as B ranges over N (Proposition 6). So $UP_{h \otimes s}U$ commutes with all \tilde{B} for B in N .

Let $V = P_{h \otimes s}U$, then $VV^* = P_{h \otimes s}$ ($= I_{h \otimes s}$). Define $V^*V = P = U^*P_{h \otimes s}U$. Then P is in $(N \otimes I_s)'$. So $\tilde{\varphi}(\tilde{A}) = V\tilde{A}V^*$ for all A in M . Claim: V is in $(N \otimes I_s)'$. Let B be in N , $\tilde{B} = \tilde{\varphi}(\tilde{B}) = V\tilde{B}V^*$ so $V^*\tilde{B} = P\tilde{B}V^* = \tilde{B}PV^* = (\tilde{B})V^*$ so V is in \tilde{N}' . Now

$$\begin{aligned} P\tilde{A}P &= V^*V\tilde{A}V^*V \\ &= V^*\tilde{\varphi}(\tilde{A})V \\ &= V^*V\tilde{\varphi}(\tilde{A}) = P\tilde{\varphi}(\tilde{A}) = \tilde{\varphi}(\tilde{A})P \text{ (as } \tilde{\varphi}(\tilde{A}) \in N \otimes I_s \text{)}. \end{aligned}$$

Now let $\tilde{A}P = 0$ ($A \geq 0$) then $\tilde{A}V^*V = 0$ so $V\tilde{A}V^*V = 0 = \tilde{\varphi}(\tilde{A})V$ so $\tilde{\varphi}(\tilde{A})P_{h \otimes s}U = 0$ and $\tilde{\varphi}(\tilde{A})P_{h \otimes s} = 0$ so $(\varphi(A) \otimes I_s)(x \otimes u) = 0$ for all x in h and u in s implies $\varphi(A) = 0$ so $A = 0$, by faithfulness of φ .

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Received November 13, 1967. This work is part of a dissertation submitted for the Ph. D. at UCLA under the direction of Dr. Dye.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

Vol. 27, No. 2

February, 1968

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