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**ON COMMUTATIVE, NONPOTENT ARCHIMEDEAN
SEMIGROUPS**

RICHARD G. LEVIN

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In this paper we will study commutative, archimedean, nonpotent (i.e., without an idempotent) semigroups, obtaining several results concerning finitely generated ones. The main theorem of this paper is the following: a finitely generated, commutative, archimedean, nonpotent semigroup is power joined. The main theorem is derived by considering the decomposition of the semigroup S into a union of disjoint semilattices; the congruence ρ_b , defined by $x\rho_b y$ if and only if there exist positive integers n and m such that $b^n x = b^m y$, determines the union, whereas congruence classes are semilattices under the partial order \geq_b defined by $x \geq_b y$ if and only if $y = b^n x$ or $y = x$. The set of maximal elements relative to \geq_b generates S . The following is a crucial lemma in the proof of the main theorem: let S be a finitely generated, commutative, nonpotent, archimedean semigroup; then the set of maximal elements of S relative to \geq_b is a finite set.

Let S be a commutative, nonpotent, archimedean semigroup. We will define a congruence ρ on S and state several results concerning S/ρ and the congruence classes of S modulo ρ . The remarks and definitions which precede Definition 5 will be used in several instances; a complete discussion can be found in [5]. See [6] and [7] for an abstract of these results. Proofs of all other results in this paper are supplied.

DEFINITION 1. Let $b \in S$. The binary relation ρ_b on S is defined by $x\rho_b y$ if and only if there exist positive integers n and m such that $b^n x = b^m y$.

The relation ρ_b is a congruence relation on S and b is called the standard element determining the corresponding decomposition of S . Furthermore, for any b , S/ρ_b is a group; the congruence class modulo ρ_b containing b is the identity element of S/ρ_b and it is a subsemigroup of S . We call S/ρ_b the structure group of S with respect to b .

DEFINITION 2. Let S_α be an arbitrary congruence class of S (mod ρ_b). The following relation, \geq_b is a partial order on S_α . Let $x, y \in S_\alpha$. We define \geq_b on S_α by $x \geq_b y$ if and only if there exists a positive integer n such that $y = b^n x$, or $y = x$.

DEFINITION 3. A discrete tree R is a lower semilattice (i.e., a

partially ordered set in which every pair of elements have a greatest lower bound) satisfying:

- (a) for all $x, y, z \in R$, $x < z$ and $y < z$ imply $x \leq y$ or $y \leq x$, and
- (b) the set $\{x \mid x \in R \text{ and } b \leq x \leq c\}$ is a finite set for any pair $b, c \in R$.

Let S_α be a congruence class of S modulo ρ_b . Then S_α is a discrete tree with respect to the partial order \geq_b .

DEFINITION 4. An element x of S is called a prime element of S relative to the congruence ρ_b if x is not divisible by b . Or, alternately, x is a prime element if x is a maximal element of a congruence class S_α of $S \pmod{\rho_b}$ relative to the partial order \geq_b defined on S_α .

The following two remarks are particularly useful.

REMARK 1. Let $a \in S$. Then

$$\bigcap_{n=1}^{\infty} a^n S = \emptyset .$$

REMARK 2. Let $a, b \in S$. Then

$$a \neq ab .$$

DEFINITION 5. Let R be an arbitrary semigroup. We define the binary relation \leq on R by $a \leq b$ if and only if there exists $x \in R$ such that $a = bx$, or $a = b$. If $a \neq b$ and $a \leq b$ we generally write $a < b$.

LEMMA 1. Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Then the relation \leq on S is a partial order and S satisfies the ascending chain condition relative to \leq .

Proof. It follows from the definition that reflexivity is satisfied. Suppose that $a, b \in S$ and $a \leq b$ and $b \leq a$. Then either $a = bx$ and $b = ay$, or $a = b$. Consider the former. We conclude that

$$(1) \quad a = (ay)x = a(yx) .$$

But (1) contradicts Remark 2. Therefore $a = b$. Thus, \leq is antisymmetric. Suppose $a \leq b$ and $b \leq c$. We suppose also that $a \neq b$ and $b \neq c$. Then $a = bx$ and $b = cy$ for some $x, y \in S$. Therefore,

$$(2) \quad a = (cy)x = c(yx) .$$

Thus, $a \leq c$, and now it is obvious that \leq is transitive.

Suppose there exists a sequence of elements of S , $\{a_n \mid n > 0\}$,

satisfying

$$(3) \quad a_1 < a_2 < a_3 < \dots < a_n \dots .$$

Let $T = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a finite generating set of S . The sequence (3) reduces to the set of equations:

$$(4) \quad a_n = a_{n+1}x_{n+1}, \quad \text{for all } n \geq 1, \text{ where } x_{n+1} \in S .$$

The set of equations (4) leads to the set

$$(5) \quad a_1 = a_2x_2 = a_3x_2x_3 = \dots = a_nx_2x_3 \dots x_n = \dots .$$

For all k , x_k can be expressed as

$$(6) \quad x_k = \alpha_1^{r_{k1}}\alpha_2^{r_{k2}} \dots \alpha_n^{r_{kn}} ,$$

and $r_{kj} \geq 0$ for $j = 1, 2, \dots, n$, and there exists $j_0, 1 \leq j_0 \leq n$ such that $r_{kj_0} > 0$. Multiplying the x_k 's, we arrive at

$$(7) \quad \begin{aligned} x_2 &= \alpha_1^{r_{21}}\alpha_2^{r_{22}} \dots \alpha_n^{r_{2n}} \\ x_2x_3 &= \alpha_1^{p_{31}}\alpha_2^{p_{32}} \dots \alpha_n^{p_{3n}} \\ x_2x_3 \dots x_k &= \alpha_1^{p_{k1}}\alpha_2^{p_{k2}} \dots \alpha_n^{p_{kn}} , \end{aligned}$$

where $0 \leq r_{2j} \leq p_{3j} \leq p_{4j} \leq \dots \leq p_{kj} \leq \dots$, for j satisfying $1 \leq j \leq n$.

If for some $j, 1 \leq j \leq n$, we have

$$(8) \quad \lim_{n \rightarrow \infty} p_{nj} = +\infty ,$$

then we can write

$$(9) \quad a_1 = \alpha_j y_1 = (\alpha_j)^2 y_2 = (\alpha_j)^3 y_3 = \dots = (\alpha_j)^k y_k = \dots ,$$

and we conclude that

$$a_1 \in \bigcap_{n=1}^{\infty} (\alpha_j)^n S .$$

This contradicts Remark 1. We set

$$(10) \quad R_j = \lim_{n \rightarrow \infty} P_{nj} ,$$

and

$$(11) \quad M = \max \{R_1, R_2, \dots, R_n\} .$$

The number M is finite and it is now obvious that there exists an integer $N \geq M$ such that

$$(12) \quad x_2x_3 \dots x_N = x_2x_3 \dots x_Nx_{N+1} .$$

This contradicts Remark 2, and the contradiction establishes that S

satisfies the ascending chain condition relative to the relation \leq .

LEMMA 2. *Let S be a commutative, nonpotent, archimedean semigroup and let \leq be the partial order on S defined above (see Definition 4). Let S satisfy the ascending chain condition relative to \leq . Then the set of maximal elements relative to \leq is a generating set for S and is contained in every other generating set of S .*

Proof. Let S' be an arbitrary generating set for S and let S'' be the set of all the maximal elements of S relative to \leq . Let $\beta \in S''$. Then

$$(12) \quad \beta = \alpha_1 \alpha_2 \cdots \alpha_n, \quad \text{where } \alpha_i \in S'$$

Suppose $n > 1$. Then $\beta = \alpha_1(\alpha_2 \cdots \alpha_n)$. This implies that $\beta < \alpha_1$. Since this is impossible, $n = 1$ and $\beta \in S'$. That is, $S'' \subseteq S'$.

Let $x \in S$. Then

$$(13) \quad x = \alpha_1 \alpha_2 \cdots \alpha_n, \quad \text{where } \alpha_i \in S' \text{ for } 1 \leq i \leq n.$$

Fix t , $1 \leq t \leq n$. Suppose α_t is not a maximal element. Then there exists $\beta_t \in S''$ such that $\alpha_t < \beta_t$ and $\alpha_t = \beta_t x_t$ for some $x_t \in S$. If x_t is not a maximal element then there exists a maximal element β_{t1} such that $x_t = \beta_{t1} x_{t1}$. By definition of \leq , $x_t < x_{t1}$. We continue in this fashion. After N steps we arrive at the equation

$$(14) \quad \alpha_t = \beta_t \beta_{t1} \beta_{t2} \cdots \beta_{t,N-1} x_{t,N-1}$$

and the sequence of inequalities

$$(15) \quad \alpha_t < x_t < x_{t1} < x_{t2} \cdots < x_{t,N-1},$$

where $\beta_{t,k}$ is a maximal element for $1 \leq k \leq N-1$. Since S satisfies the ascending chain condition, we conclude that this procedure must lead to a maximal element $x_{t,M}$ in (14) and (15). Setting $M = N-1$ in (14) and substituting (14) into (13) we express x as a product of maximal elements. We conclude that S'' generates S and that S'' is the smallest generating set of S .

PROPOSITION 3. *Let S be a finitely generated, commutative, nonpotent, archimedean semigroup. Then the set of maximal elements (relative to \leq) of S is a finite set.*

Proof. According to Lemma 1, S satisfies the ascending chain condition relative to \leq . By Lemma 2, the set of maximal elements of S is a subset of every generating set of S . Since S is finitely generated, the set of maximal elements of S must be a finite set.

PROPOSITION 4. Let S be a finitely generated, commutative, non-potent, archimedean semigroup. Let $a \in S$. Then the set of prime elements of S , with respect to the standard element a , is a finite set.

Proof. Suppose there are an infinite number of primes. Let $S' = \{b_1, b_2, \dots, b_k, \dots\}$ be a countably infinite subset of the set of primes of S . Let $T = \{a_1, a_2, \dots, a_n\}$ be the set of all maximal elements of S . Every element of S' admits a representation of the form

$$(16) \quad b_j = a_1^{\mu_{j1}} a_2^{\mu_{j2}} \dots a_i^{\mu_{ji}} \dots a_n^{\mu_{jn}}, \mu_{ji} \geq 0 .$$

Consider the sequences

$$(17) \quad \mu_{1i}, \mu_{2i}, \mu_{3i}, \dots, \mu_{ki}, \dots, \text{ where } 1 \leq i \leq n .$$

For at least one i between 1 and n the corresponding sequence (17) will be unbounded. Otherwise we immediately conclude that S' is a finite set. Suppose the sequence for i_0 is unbounded. Choose a subsequence

$$(18) \quad \mu_{t_1, i_0}, \mu_{t_2, i_0}, \dots, \mu_{t_r, i_0}, \dots ,$$

satisfying

$$(19) \quad \mu_{t_1, i_0} < \mu_{t_2, i_0} < \dots .$$

For convenience, we will change the notation. Set

$$(20) \quad r_j = \mu_{t_j, i_0}, \quad \text{for } j \geq 1 .$$

We now have

$$(21) \quad b_{j_k} = a_1^{r_k} \dots a_{i_0}^{r_k} \dots a_n^{r_k}, \quad \text{for } k \geq 1 .$$

For all $k, b_{j_k} \neq ax_k$ for any $x_k \in S$. But since S is an archimedean semigroup, there exists an integer $l, l > 0$ such that

$$(22) \quad (a_{i_0})^l = a\mu .$$

There exists $k_0 > 0$ such that $r_k > l$ for all $k \geq k_0$. Therefore we have

$$(23) \quad \begin{aligned} b_{j_k} &= a_1^{r_k} \dots a_{i_0}^{r_k} \dots a_n^{r_k} \\ &= a_1^{r_k} \dots a_{i_0}^l a_{i_0}^{r_k-l} \dots a_n^{r_k} \\ &= a\mu(a_1^{r_k-l} \dots a_{i_0}^{r_k-l} \dots a_n^{r_k-l}), \quad \text{for all } k \geq k_0 \end{aligned}$$

This contradiction establishes that the set of primes is a finite set.

PROPOSITION 5. Let S be a finitely generated, commutative, non-potent, archimedean semigroup. Let ρ_b be the congruence relation of Definition 1. Then S/ρ_b is a finite group.

Proof. The elements of S/ρ_b are congruence classes. Each congruence class is a tree and contains at least one prime element. If S/ρ_b were an infinite group, then S would contain an infinite number of prime elements. This would contradict Proposition 4. Thus S/ρ_b is a finite group.

THEOREM 1. *A finitely generated, commutative, nonpotent, archimedean semigroup is power joined.*

Proof. Let $a \in S$. Consider S/ρ_a . Let $x, y \in S$. Let $\alpha, \beta \in S/\rho_a$ such that $x \in S_\alpha$ and $y \in S_\beta$. Since S/ρ_a is a finite group, there exist positive integers n and m such that $\alpha^n = \varepsilon, \beta^m = \varepsilon$, where ε is the identity of S/ρ_a . Therefore, $x^n \in S_\varepsilon$ and $y^m \in S_\varepsilon$. Let $\{P_1, P_2, \dots, P_r\}$ be the set of prime elements of S which are contained in S_ε . Let

$$P = \text{glb}\{P_1, P_2, \dots, P_r\},$$

where the partial order in S_ε is \geq_a (see Definition 1). Set

$$T = \{Z \mid Z \in S_\varepsilon, Z \geq_a P\}.$$

T is a finite set because S_ε is a discrete tree. Since S is nonpotent, the powers of x^n and y^m are all distinct. Therefore there exist positive integers r, t such that $P \geq_a (x^n)^r$ and $P \geq_a (y^m)^t$. Further, for all Z , where $P \geq_a Z$, there exists a positive integer s such that $Z = a^s$. Therefore,

$$(24) \quad (x^n)^r = a^{\mu_1}, (y^m)^t = a^{\nu_1},$$

and

$$(25) \quad (x^{nr})^{\nu_1} = (y^{mt})^{\mu_1}.$$

We conclude that S is a power joined semigroup.

THEOREM 2. *Let S be a commutative, nonpotent, archimedean semigroup. Let $a \in S$ and let $G_a (= S/\rho_a)$ be the corresponding structure group. Then, S is power joined if and only if G_a is a periodic group and S_ε is power joined (where S_ε is the congruence class of $S \bmod \rho_a$ which contains a).*

Proof. Let S be power joined. Let $\alpha \in G_a, y \in S_a$. There exist positive integers n and m such that $y^n = a^m$. Since $a^m \in S_\varepsilon$, so is y^n . Therefore $\alpha^n = \varepsilon$ and we conclude that G_a is periodic. The set S_ε is power joined because it is a subsemigroup of S .

To prove the converse, let $x, y \in S$. There exist $\alpha, \beta \in G_a$ such that $x \in S_\alpha, y \in S_\beta$ and $\alpha^n = \varepsilon, \beta^m = \varepsilon$. Therefore, $x^n \in S_\varepsilon, y^m \in S_\varepsilon$. Since

S_c is power joined, there exist positive integers k and l such that

$$(26) \quad (x^n)^k = (y^m)^l.$$

We conclude that S is power joined.

THEOREM 3. *Let S be a commutative, nonpotent, archimedean semigroup. Then S is power joined if and only if every finitely generated subsemigroup is archimedean.*

Proof. Let S be power joined. Let S' be a finitely generated subsemigroup of S . Then S' is also power joined. Let $x, y \in S'$. Then there exist positive integers n, m such that $x^n = y^m$. Set $\mu = y^{m-1}$, $v = x^{n-1}$. We get $x^n = y\mu$ and $y^m = xv$. The elements μ and v are also in S' . In case n or m equals 1, we can easily arrange the desired equations by multiplying both sides of the equation $x^n = y^m$ by x or y as required. Therefore S' is archimedean.

Let $x, y \in S$. Let S' be the subsemigroup of S generated by x and y . Since S' is finitely generated, it is archimedean. Thus, S' is a finitely generated, commutative, nonpotent, archimedean semigroup, and by Theorem 1 we conclude that S' is power joined. Therefore, there exist positive integers n and m such that $x^n = y^m$. Since x and y were arbitrary elements of S , we now conclude that S is power joined.

REFERENCES

1. James Christlock, *The structure of archimedean semigroups*, Thesis, University of California, Davis, 1966.
2. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Volume 1, Amer. Math. Soc., Providence, R. I., 1961.
3. Mario Petrich, *On the structure of a class of commutative semigroups*, Czechoslovak Math. J. (89), **14** (1964) Prague.
4. Takayuki Tamura, *Commutative, nonpotent, archimedean semigroups with cancellation law 1*, J. of Gakugei, Tokushima Univ. **8** (1957), 5-11.
5. ———, *Construction of commutative, archimedean semigroups*, (to be published in Math. Nachr., Germany)
6. ———, *Notes on commutative, archimedean semigroups I*, Proc. Japan Acad. **42** (1966), 35-40.
7. ———, *Notes on commutative, archimedean semigroup II*, Proc. Japan Acad. **42** (1966), 545-548.

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