FUNCTIONS REPRESENTED BY RADEMACHER SERIES

JAMES R. McLAUGHLIN
A series of the form \( \sum_{m=1}^{\infty} a_m r_m(t) \), where \( \{a_m\} \) is a sequence of real numbers and \( r_m(t) \) denotes the \( m \)th Rademacher function, \( \text{sign } \sin(2^n \pi t) \), is called a Rademacher series (as usual, \( \text{sign } 0 = 0 \)).

Letting \( f(t) \) denote the sum of this series whenever it exists, we shall investigate the effect that various conditions on \( \{a_m\} \) have on the continuity, variation, and differentiability properties of \( f \).

2. Continuity properties. We now prove

**Theorem (2.1).** If \( \sum |a_m| < \infty \), then \( f(t) \) is continuous at dyadic irrationals (i.e., numbers not of the form \( p/2^k \)) and has right and left hand limits everywhere in \([0, 1] \).

**Proof.** Under our hypothesis we have that \( \sum a_m r_m(t) \) converges uniformly to \( f(t) \), which implies our conclusion since the Rademacher functions are continuous at dyadic irrationals and have right and left hand limits everywhere in \([0, 1] \).

In general, the right and left hand limits of \( f(t) \) are unequal at dyadic rationals. We now investigate under what conditions we have equality and prove.

**Theorem (2.2).** If \( \sum |a_m| < \infty \), then the following are equivalent:

(a) \( a_k = \sum_{m=k+1}^{\infty} a_m \),
(b) \( f(p2^{-k} + \varepsilon_n) \to f(p2^{-k}) \) as \( n \to \infty \),
(c) \( f(p2^{-k} + \delta_n) \to f(p2^{-k}) \) as \( n \to \infty \),
(d) \( f(p2^{-k} + \varepsilon_n) - f(p2^{-k} + \delta_n) \to 0 \) as \( n \to \infty \),

where \( \{\varepsilon_n\} \) and \( \{\delta_n\} \) are some positive and negative sequences tending to zero, and \( p \) is an odd integer.

**Proof.**

\[
\begin{align*}
f(p2^{-k} + t) - f(p2^{-k}) &= \sum_{m=k+1}^{\infty} a_m r_m(p2^{-k} + t) - a_k r_k(t) \\
&\quad + \sum_{m=k+1}^{\infty} a_m r_m(t) - \sum_{m=k+1}^{k-1} a_m r_m(p2^{-k}) ,
\end{align*}
\]

since \( r_m(p2^{-k} + t) = r_m(t) \) if \( m \geq k + 1 \), and \( r_k(p2^{-k} + t) = -r_k(t) \).
Therefore,
\[ f(p2^{-k} + e_n) - f(p2^{-k}) \to -a_k + \sum_{m=k+1}^{\infty} a_m \quad \text{as} \quad n \to \infty . \]

This shows the equivalence of (a) and (b). A similar argument establishes the equivalence of (a), (c), and (d).

We have, at once, the following

**Corollary (2.1).** For absolutely convergent Rademacher series the following are equivalent:

(i) \( f(t) \) is continuous at \( p2^{-k} \) for some odd integer \( p \),
(ii) \( f(t) \) is continuous at \( p2^{-k} \) for all odd integers \( p \),
(iii) \( a_k = \sum_{m=k+1}^{\infty} a_m \).

**Remarks.**

1. Notice that, if \( a_k = \sum_{m=k+1}^{\infty} a_m \) and \( a_{k+1} = \sum_{m=k+2}^{\infty} a_m \), then \( a_{k+1} = (a_k)/2 \).

2. Theorem (2.2) is false under the hypothesis that \( \sum |a_m| = \infty \) and \( a_m \to 0 \), since under these conditions we have that in every interval \( f(t) \) assumes every real number \( c \) times [2, p. 234, Th. 2].

This shows that the existence of the limit in the sense of Theorem (2.2) implies no relationship whatever between \( a_k \) and \( \sum_{m=k+1}^{\infty} a_m \). Also by choosing \( \{a_m\} \) such that \( \sum (a_m)^2 = \infty \) we see that the existence of the limit in the above sense does not even imply that \( \sum a_m r_m(t) \) converges in a set of positive measure [8, p. 212].

3. If \( f(t) = \sum a_m r_m(t) \) is essentially bounded, then \( \sum |a_m| < \infty \) (see [3]).

We now omit the condition that \( \sum |a_m| < \infty \) and prove

**Theorem (2.3)** \( a_k = (a_{k-1})/2, k > 1 \), if either

\[
\lim_{n \to \infty} [f(2^{-k} + p2^{-k+2} + e_n) - f(2^{-k+1} + p2^{-k+2} + e_n)]
\]

or

\[
\lim_{n \to \infty} [f(2^{-k+1} + p2^{-k+2} + e_n)]
\]

where \( e_n > 0, \delta_n < 0, \lim e_n = \lim \delta_n = 0 \) and \( p \) is an integer.

**Proof.** If \( k > 1 \), \( f(t) \)
functions represented by rademacher series

\[ f(2^{-k} + p2^{-k+2} + t) = f(2^{-k+1} + p2^{-k+2} + t) \]

\[ = a_k \left[ r_k(2^{-k} + p2^{-k+2} + t) - r_{k-1}(2^{-k+1} + p2^{-k+2} + t) \right] + \cdots \]

\[ + a_{k-1}[r_{k-1}(2^{-k} + p2^{-k+2} + t) - r_{k-2}(2^{-k+1} + p2^{-k+2} + t)] \]

\[ + \ldots + a_1[r_1(2^{-k} + p2^{-k+2} + t) - r_0(t)] + a_0[-r_0(t) - r_1(t)] . \]

Thus,

\[ \lim_{n \to \infty} A_{\varepsilon_n} = 2a_{k-1} - 2a_k \quad \text{and} \quad \lim_{n \to \infty} A(\delta_n) = 2a_k . \]

In view of (1) we have then \( 2a_k = a_{k-1} \).

A similar proof will suffice if equation (2) is valid.

Remark. In much the same way we can prove a more general result, namely that if \( \{c_k\} \) has the property that

\[ \sum_{m=1}^{\infty} 1/ \Pi_{k=1}^{m} (1 + c_k) = c^{-1} \neq 0 \]

is absolutely convergent, then

\[ f(t) = cf(0+\sum_{m=1}^{\infty} r_m(t)/ \Pi_{k=1}^{m} (1 + c_k) \]

if and only if for every \( k > 1 \) we have that in (1) the first limit equals \( c_k \) times the second.

We now utilize the concepts of approximate limits and approximately continuous functions (see [5, pp. 132, 219]). From Theorem (2.3), we deduce immediately.

Corollary 2.2. If the approximate limit of \( f(t) \) exists at either \( 2^{-k} + p2^{-k+2} \) and \( 2^{-k+1} + p2^{-k+2} \) or \( 2^{-k+1} + p2^{-k+2} \) and \( 3 \cdot 2^{-k} + p2^{-k+2} \) (where \( k > 1 \) and \( p \) is any integer), then \( a_k = (a_{k-1})/2 \).

We now prove

Corollary (2.3). If \( F(t) \) is approximately continuous in \([0, 1]\) and \( \sum a_m r_m(t) \) converges a.e. in \([0, 1]\) to \( F(t) \), then

\[ F(t) = F(0) \cdot (1 - 2t), \quad a_m = F(0)/2^m (m = 1, 2, \ldots) . \]

Proof. Since \( F(t) \) is approximately continuous in \([0, 1]\), we have that \( f(t) \) has approximate limits everywhere. Thus

\[ F(t) = C \sum r_m(t)/2^m \quad \text{a.e.,} \quad C \text{ being a constant.} \]

But, since \( \sum r_m(t)/2^m = 1 - 2t \) a.e. (see [7, p. 220]), this implies that

\[ F(t) = C(1 - 2t) \quad \text{a.e.} \]
which concludes our proof since $F(t)$ is approximately continuous.

Remarks. 1. Corollary (2.2) shows that, if the approximate limits of $f(t)$ exist at certain dyadic rationals, then $a_m = C/2^m$ for $m \geq m_0$ (where $m_0, C$ are constants).

2. The conclusion of Corollary (2.3) was proved by Wang Si-Lei ([6, p. 704]; cf. [7, p. 221]) under the stronger hypothesis that $F(t)$ be continuous in $[0,1]$. Wang's result can also be obtained from Theorem (2.2) and Remarks (1) and (3) following it.

3. Corollary (2.2) is a generalization of some theorems of Wang [6, Th. 1, 2, 3].

4. In Corollary (2.3), the condition "convergent a.e." cannot be replaced by "convergent in $E \subset [0,1], |E| < 1" [6, p. 706].

3. Variational properties. A. I. Rubinstein has shown [4, p. 143] that if $\sum |a_m| 2^m < \infty$, then $f(t) \in \text{Lip}(1,1)$.

In order to strengthen this result we now state the following lemma which follows from Minkowski's inequality:

Lemma (3.1). If $V_p(f_m)$ denotes the $p$th variation of $f_m(t)$, then

(i) if $0 < p \leq 1$, $V_p^p \left( \sum_{m=1}^{\infty} f_m \right) \leq \sum_{m=1}^{\infty} V_p^p(f_m)$;

(ii) if $p \geq 1$, $V_p \left( \sum_{m=1}^{\infty} f_m \right) \leq \sum_{m=1}^{\infty} V_p(f_m)$.

We will now prove

Theorem (3.1). (i) If $0 < p \leq 1$, then $\sum |a_m| 2^m < \infty$ implies $f(t)$ is of bounded $p$th variation;

(ii) if $p \geq 1$, then $\sum |a_m| 2^{m/p} < \infty$ implies $f(t)$ is of bounded $p$th variation;

(iii) if $0 < p \leq 1$, then $a_m \downarrow 0, \sum a_m^p 2^{m} = \infty$ implies $g(t) = \sum (-1)^m a_m r_m(t)$

is not of bounded $p$th variation.

Proof. Parts (i) and (ii) are immediate by the lemma.

Also, setting $\{t_i\} = \{2^{-s-1} + i 2^{-s}\}_{i=0}^{s-1}$ and $b_m = (-1)^m a_m$ we obtain

$$\sum_{i=1}^{s} \left| g(t) - g(t_{i-1}) \right| = \left| -2b_1 + \cdots + 2b_s \right|^p$$

$$+ 2 \left| -2b_2 + \cdots + 2b_s \right|^p + \cdots + 2^{s-2} \left| -2b_{s-1} + 2b_s \right|^p$$

$$+ 2^{s-1} \left| 2b_s \right|^p \geq \sum_{i=1}^{s} 2^{i-1} \left| 2b_i \right|^p \to \infty \text{ as } n \to \infty .$$
4. Differentiability properties. With regard to differentiability, L. A. Balasov has shown [1, p. 631] that \( f(t) \) has a derivative at least one point if and only if

\[
\lim_{n \to \infty} 2^n a_n = A \text{ exists}.
\]

Balasov has demonstrated that this condition alone is not sufficient in order to have \( f(t) \) differentiable a.e. [1, pp. 633-4]. He then proves that condition (3) and the relation

\[
a_k \geq \sum_{m=k+1}^{\infty} a_m \quad \text{for every } k \geq 1
\]

implies \( f(t) \) is monotone in \([0,1]\), which of course implies differentiability almost everywhere.

We now prove

**Theorem (4.1).** (i) If \( \sum |a_m| 2^m < \infty \), then \( f(t) \) is differentiable almost everywhere;

(ii) if \( \{\varepsilon_n\} \) is any null sequence, then there exists a sequence \( \{a_m\} \) satisfying

(a) \( \sum a_n 2^m \varepsilon_m < \infty \),

(b) \( f(t) = \sum a_n r_n(t) \) is differentiable nowhere.

**Proof.** Part (i) follows immediately from Theorem (3.1).

Part (ii). Since \( \{\varepsilon_n\} \) is a null sequence, there exists an increasing sequence of positive integers \( \{N_m\} \) such that

\[
\varepsilon_{N_m} < 2^{-m}, \quad m = 1, 2, \ldots
\]

Now set

\[
a_m = 2^{-m}, \quad i = 2, 4, 6, \ldots
\]

Then (a) follows from condition (4), and (b) follows since Balasov's condition (3) for differentiability is not satisfied.

**Remark.** It would be interesting to know if the sum, \( f(t) \), of a Rademacher series is of bounded variation whenever \( f(t) \) is differentiable almost everywhere (as is the case for lacunary trigonometric series).
REFERENCES


Received June 27, 1967. This research was supported by a National Aeronautics and Space Administration Fellowship.

WAYNE STATE UNIVERSITY AND PENNSYLVANIA STATE UNIVERSITY
Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsuusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leonard E. Baum and George Roger Sell, <em>Growth transformations for functions on manifolds</em></td>
<td>211</td>
</tr>
<tr>
<td>Henry Gilbert Bray, <em>A note on CLT groups</em></td>
<td>229</td>
</tr>
<tr>
<td>Douglas Napier Clark, <em>On matrices associated with generalized interpolation problems</em></td>
<td>241</td>
</tr>
<tr>
<td>Richard Brian Darst and Euline Irwin Green, <em>On a Radon-Nikodym theorem for finitely additive set functions</em></td>
<td>255</td>
</tr>
<tr>
<td>Carl Louis DeVito, <em>A note on Eberlein’s theorem</em></td>
<td>261</td>
</tr>
<tr>
<td>P. H. Doyle, III and John Gilbert Hocking, <em>Proving that wild cells exist</em></td>
<td>265</td>
</tr>
<tr>
<td>Leslie C. Glaser, <em>Uncountably many almost polyhedral wild (k – 2)-cells in $E^k$ for $k \geq 4$</em></td>
<td>267</td>
</tr>
<tr>
<td>Samuel Irving Goldberg, <em>Totally geodesic hypersurfaces of Kaehler manifolds</em></td>
<td>275</td>
</tr>
<tr>
<td>Donald Goldsmith, <em>On the multiplicative properties of arithmetic functions</em></td>
<td>283</td>
</tr>
<tr>
<td>Jack D. Gray, <em>Local analytic extensions of the resolvent</em></td>
<td>305</td>
</tr>
<tr>
<td>Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, <em>Commutativity theorems for nonassociative rings with a finite division ring homomorphic image</em></td>
<td>325</td>
</tr>
<tr>
<td>André (Piotrowsky) De Korvin, <em>Normal expectations in von Neumann algebras</em></td>
<td>333</td>
</tr>
<tr>
<td>James Donald Kuelbs, <em>A linear transformation theorem for analytic Feynman integrals</em></td>
<td>339</td>
</tr>
<tr>
<td>W. Kuich, <em>Quasi-block-stochastic matrices</em></td>
<td>353</td>
</tr>
<tr>
<td>Richard G. Levin, <em>On commutative, nonpotent archimedean semigroups</em></td>
<td>365</td>
</tr>
<tr>
<td>James R. McLaughlin, <em>Functions represented by Rademacher series</em></td>
<td>373</td>
</tr>
<tr>
<td>Calvin R. Putnam, <em>Singular integrals and positive kernels</em></td>
<td>379</td>
</tr>
<tr>
<td>Harold G. Rutherford, II, <em>Characterizing primes in some noncommutative rings</em></td>
<td>387</td>
</tr>
<tr>
<td>Benjamin L. Schwartz, <em>On interchange graphs</em></td>
<td>393</td>
</tr>
<tr>
<td>Satish Shirali, <em>On the Jordan structure of complex Banach$^</em>$algebras*</td>
<td>397</td>
</tr>
<tr>
<td>Earl J. Taft, <em>A counter-example to a fixed point conjecture</em></td>
<td>405</td>
</tr>
<tr>
<td>J. Roger Teller, <em>On abelian pseudo lattice ordered groups</em></td>
<td>411</td>
</tr>
</tbody>
</table>