

Pacific Journal of Mathematics

FUNCTIONS REPRESENTED BY RADEMACHER SERIES

JAMES R. MCLAUGHLIN

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A series of the form $\sum_{m=1}^{\infty} a_m r_m(t)$, where $\{a_m\}$ is a sequence of real numbers and $r_m(t)$ denotes the m th Rademacher function, $\text{sign } \sin(2^m \pi t)$, is called a Rademacher series (as usual, $\text{sign } 0 = 0$).

Letting $f(t)$ denote the sum of this series whenever it exists, we shall investigate the effect that various conditions on $\{a_m\}$ have on the continuity, variation, and differentiability properties of f .

2. Continuity properties. We now prove

THEOREM (2.1). *If $\sum |a_m| < \infty$, then $f(t)$ is continuous at dyadic irrationals (i.e., numbers not of the form $p/2^k$) and has right and left hand limits everywhere in $[0, 1]$.*

Proof. Under our hypothesis we have that $\sum a_m r_m(t)$ converges uniformly to $f(t)$, which implies our conclusion since the Rademacher functions are continuous at dyadic irrationals and have right and left hand limits everywhere in $[0, 1]$.

In general, the right and left hand limits of $f(t)$ are unequal at dyadic rationals. We now investigate under what conditions we have equality and prove.

THEOREM (2.2). *If $\sum |a_m| < \infty$, then the following are equivalent:*

- (a) $a_k = \sum_{m=k+1}^{\infty} a_m$,
- (b) $f(p2^{-k} + \varepsilon_n) \rightarrow f(p2^{-k})$ as $n \rightarrow \infty$,
- (c) $f(p2^{-k} + \delta_n) \rightarrow f(p2^{-k})$ as $n \rightarrow \infty$,
- (d) $f(p2^{-k} + \varepsilon_n) - f(p2^{-k} + \delta_n) \rightarrow 0$ as $n \rightarrow \infty$,

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are some positive and negative sequences tending to zero, and p is an odd integer.

Proof.

$$f(p2^{-k} + t) - f(p2^{-k}) = \sum_{m=1}^{k-1} a_m r_m(p2^{-k} + t) - a_k r_k(t) \\ + \sum_{m=k+1}^{\infty} a_m r_m(t) - \sum_{m=1}^{k-1} a_m r_m(p2^{-k}),$$

since $r_m(p2^{-k} + t) = r_m(t)$ if $m \geq k + 1$, and $r_k(p2^{-k} + t) = -r_k(t)$.

Therefore,

$$f(p2^{-k} + \varepsilon_n) - f(p2^{-k}) \rightarrow -a_k + \sum_{m=k+1}^{\infty} a_m \text{ as } n \rightarrow \infty .$$

This shows the equivalence of (a) and (b). A similar argument establishes the equivalence of (a), (c), and (d).

We have, at once, the following

COROLLARY (2.1). *For absolutely convergent Rademacher series the following are equivalent:*

- (i) $f(t)$ is continuous at $p2^{-k}$ for some odd integer p ,
- (ii) $f(t)$ is continuous at $p2^{-k}$ for all odd integers p ,
- (iii) $a_k = \sum_{m=k+1}^{\infty} a_m .$

REMARKS. 1. Notice that, if $a_k = \sum_{m=k+1}^{\infty} a_m$ and $a_{k+1} = \sum_{m=k+2}^{\infty} a_m$, then $a_{k+1} = (a_k)/2$.

2. Theorem (2.2) is false under the hypothesis that $\sum |a_m| = \infty$ and $a_m \rightarrow 0$, since under these conditions we have that in every interval $f(t)$ assumes every real number c times [2, p. 234, Th. 2].

This shows that the existence of the limit in the sense of Theorem (2.2) implies no relationship whatever between a_k and $\sum_{m=k+1}^{\infty} a_m$. Also by choosing $\{a_m\}$ such that $\sum (a_m)^2 = \infty$ we see that the existence of the limit in the above sense does not even imply that $\sum a_m r_m(t)$ converges in a set of positive measure [8, p. 212].

3. If $f(t) = \sum a_m r_m(t)$ is essentially bounded, then $\sum |a_m| < \infty$ (see [3]).

We now omit the condition that $\sum |a_m| < \infty$ and prove

THEOREM (2.3) $a_k = (a_{k-1})/2, k > 1, \text{ if either}$

$$(1) \quad \begin{aligned} & \lim_{n \rightarrow \infty} [f(2^{-k} + p2^{-k+2} + \varepsilon_n) - f(2^{-k+1} + p2^{-k+2} + \varepsilon_n)] \\ & = \lim_{n \rightarrow \infty} [f(2^{-k} + p2^{-k+2} + \delta_n) - f(2^{-k+1} + p2^{-k+2} + \delta_n)] \end{aligned}$$

or

$$(2) \quad \begin{aligned} & \lim_{n \rightarrow \infty} [f(2^{-k+1} + p2^{-k+2} + \varepsilon_n) - f(3 \cdot 2^{-k} + p2^{-k+2} + \varepsilon_n)] \\ & = \lim_{n \rightarrow \infty} [f(2^{-k+1} + p2^{-k+2} + \delta_n) - f(3 \cdot 2^{-k} + p2^{-k+2} + \delta_n)] \end{aligned}$$

where $\varepsilon_n > 0, \delta_n < 0, \lim \varepsilon_n = \lim \delta_n = 0$ and p is an interger.

Proof. If $k > 1, \Delta(t)$

$$\begin{aligned} &\equiv f(2^{-k} + p2^{-k+2} + t) - f(2^{-k+1} + p2^{-k+2} + t) \\ &= a_1[r_1(2^{-k} + p2^{-k+2} + t) - r_1(2^{-k+1} + p2^{-k+2} + t)] + \dots \\ &\quad + a_{k-2}[r_{k-2}(2^{-k} + p2^{-k+2} + t) - r_{k-2}(2^{-k+1} + p2^{-k+2} + t)] \\ &\quad + a_{k-1}[r_{k-1}(2^{-k} + t) + r_{k-1}(t)] + a_k[-r_k(t) - r_k(t)]. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} \Delta(\varepsilon_n) = 2a_{k-1} - 2a_k \quad \text{and} \quad \lim_{n \rightarrow \infty} \Delta(\delta_n) = 2a_k.$$

In view of (1) we have then $2a_k = a_{k-1}$.

A similar proof will suffice if equation (2) is valid.

REMARK. In much the same way we can prove a more general result, namely that if $\{c_k\}$ has the property that

$$\sum_{m=1}^{\infty} 1/\prod_{k=1}^m (1 + c_k) = c^{-1} \neq 0$$

is absolutely convergent, then

$$f(t) = cf(0+) \sum_{m=1}^{\infty} r_m(t) / \prod_{k=1}^m (1 + c_k)$$

if and only if for every $k > 1$ we have that in (1) the first limit equals c_k times the second.

We now utilize the concepts of approximate limits and approximately continuous functions (see [5, pp. 132, 219]). From Theorem (2.3), we deduce immediately.

COROLLARY 2.2. *If the approximate limit of $f(t)$ exists at either $2^{-k} + p2^{-k+2}$ and $2^{-k+1} + p2^{-k+2}$ or $2^{-k+1} + p2^{-k+2}$ and $3 \cdot 2^{-k} + p2^{-k+2}$ (where $k > 1$ and p is any integer), then $a_k = (a_{k-1})/2$.*

We now prove

COROLLARY (2.3). *If $F(t)$ is approximately continuous in $[0, 1]$ and $\sum a_m r_m(t)$ converges a.e. in $[0, 1]$ to $F(t)$, then*

$$F(t) = F(0) \cdot (1 - 2t), \quad a_m = F(0)/2^m (m = 1, 2, \dots).$$

Proof. Since $F(t)$ is approximately continuous in $[0, 1]$, we have that $f(t)$ has approximate limits everywhere. Thus

$$F(t) = C \sum r_m(t)/2^m \text{ a.e., } C \text{ being a constant.}$$

But, since $\sum r_m(t)/2^m = 1 - 2t$ a.e. (see [7, p. 220]), this implies that

$$F(t) = C(1 - 2t) \text{ a.e.}$$

which concludes our proof since $F(t)$ is approximately continuous.

REMARKS. 1. Corollary (2.2) shows that, if the approximate limits of $f(t)$ exist at certain dyadic rationals, then $a_m = C/2^m$ for $m \geq m_0$ (where m_0, C are constants).

2. The conclusion of Corollary (2.3) was proved by Wang Si-Lei ([6, p. 704]; cf. [7, p. 221]) under the stronger hypothesis that $F(t)$ be continuous in $[0, 1]$. Wang's result can also be obtained from Theorem (2.2) and Remarks (1) and (3) following it.

3. Corollary (2.2) is a generalization of some theorems of Wang [6, Th. 1, 2, 3].

4. In Corollary (2.3), the condition "convergent a.e." cannot be replaced by "convergent in $E \subset [0, 1], |E| < 1$ " [6, p. 706].

3. **Variational properties.** A. I. Rubinstein has shown [4, p. 143] that if $\sum |a_m| 2^m < \infty$, then $f(t) \in \text{Lip}(1, 1)$.

In order to strengthen this result we now state the following lemma which follows from Minkowski's inequality:

LEMMA (3.1). *If $V_p(f_m)$ denotes the p th variation of $f_m(t)$, then*

$$(i) \text{ if } 0 < p \leq 1, V_p\left(\sum_{m=1}^{\infty} f_m\right) \leq \sum_{m=1}^{\infty} V_p(f_m);$$

$$(ii) \text{ if } p \geq 1, V_p\left(\sum_{m=1}^{\infty} f_m\right) \leq \sum_{m=1}^{\infty} V_p(f_m).$$

We will now prove

THEOREM (3.1). (i) *If $0 < p \leq 1$, then $\sum |a_m|^p 2^m < \infty$ implies $f(t)$ is of bounded p th variation;*

(ii) *if $p \geq 1$, then $\sum |a_m| 2^{m/p} < \infty$ implies $f(t)$ is of bounded p th variation;*

(iii) *if $0 < p \leq 1$, then $a_m \downarrow 0, \sum a_m^p 2^m = \infty$ implies*

$$g(t) = \sum (-1)^m a_m r_m(t)$$

is not of bounded p th variation.

Proof. Parts (i) and (ii) are immediate by the lemma.

Also, setting $\{t_i\} = \{2^{-n-1} + i2^{-n}\}_{i=0}^{2^n-1}$ and $b_m = (-1)^m a_m$ we obtain

$$\begin{aligned} \sum_{i=1}^{2^n-1} |g(t_i) - g(t_{i-1})| &= |-2b_1 + \dots + 2b_n|^p \\ &+ 2|-2b_2 + \dots + 2b_n|^p + \dots + 2^{n-2}|-2b_{n-1} + 2b_n|^p \\ &+ 2^{n-1}|2b_n|^p \geq \sum_{i=1}^n 2^{i-1}|2b_i|^p \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

This demonstrates Part (iii).

4. **Differentiability properties.** With regard to differentiability, L. A. Balasov has shown [1, p. 631] that $f(t)$ has a derivative at least one point if and only if

$$(3) \quad \lim 2^m a_m = A \text{ exists.}$$

Balasov has demonstrated that this condition alone is not sufficient in order to have $f(t)$ differentiable a.e. [1, pp. 633-4]. He then proves that condition (3) and the relation

$$a_k \geq \sum_{m=k+1}^{\infty} a_m \text{ for every } k \geq 1$$

implies $f(t)$ is monotone in $[0, 1]$, which of course implies differentiability almost everywhere.

We now prove

THEOREM (4.1). (i) *If $\sum |a_m| 2^m < \infty$, then $f(t)$ is differentiable almost everywhere;*

(ii) *if $\{\varepsilon_m\}$ is any null sequence, then there exists a sequence $\{a_m\}$ satisfying*

$$(a) \quad \sum |a_m 2^m \varepsilon_m| < \infty,$$

$$(b) \quad f(t) = \sum a_m r_m(t) \text{ is differentiable nowhere.}$$

Proof. Part (i) follows immediately from Theorem (3.1).

Part (ii). Since $\{\varepsilon_m\}$ is a null sequence, there exists an increasing sequence of positive integers $\{N_m\}$ such that

$$(4) \quad |\varepsilon_{N_m}| < 2^{-m}, \quad m = 1, 2, \dots$$

Now set

$$\begin{aligned} a_m &= 2^{-m}, \text{ if } m = N_i, & i &= 2, 4, 6, \dots \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then (a) follows from condition (4), and (b) follows since Balasov's condition (3) for differentiability is not satisfied.

REMARK. It would be interesting to know if the sum, $f(t)$, of a Rademacher series is of bounded variation whenever $f(t)$ is differentiable almost everywhere (as is the case for lacunary trigonometric series).

REFERENCES

1. L. A. Balasov, *On series with gaps* (Russian), *Izv. Akad. Nauk SSSR Ser. Math.* **29** (1965), 631-644.
2. S. Kaczmarz and H. Steinhaus, *Le systeme orthogonal de M. Rademacher*, *Studia Math.* **2** (1930), 231-247.
3. F. R. Keogh, *On Rademacher series with bounded sums*, *J. London Math. Soc.* **33** (1958), 454-455.
4. A. I. Rubenstein, *On gap series* (Russian), *Izv. Uceb. Zaved. Mat.* **34** (1963), 137-148.
5. S. Saks, *Theory of the Integral*, Dover, New York, 1964.
6. Wang Si-Lei, *On the functions represented by Rademacher series*, *Chinese Math.* **4** (1963), 703-708=*Acta Math. Sinica* **13** (1963), 647-652.
7. S. B. Stechkin and P. L. Ul'janov, *On uniqueness sets* (Russian), *Izv. Akad. Nauk SSSR Ser. Math.* **26** (1962), 211-222.
8. A. Zygmund, *Trigonometric Series*, Vol. 1, Cambridge, New York, 1959.

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Leonard E. Baum and George Roger Sell, <i>Growth transformations for functions on manifolds</i>	211
Henry Gilbert Bray, <i>A note on CLT groups</i>	229
Paul Robert Chernoff, Richard Anthony Rasala and William Charles Waterhouse, <i>The Stone-Weierstrass theorem for valuable fields</i>	233
Douglas Napier Clark, <i>On matrices associated with generalized interpolation problems</i>	241
Richard Brian Darst and Euline Irwin Green, <i>On a Radon-Nikodym theorem for finitely additive set functions</i>	255
Carl Louis DeVito, <i>A note on Eberlein's theorem</i>	261
P. H. Doyle, III and John Gilbert Hocking, <i>Proving that wild cells exist</i>	265
Leslie C. Glaser, <i>Uncountably many almost polyhedral wild $(k - 2)$-cells in E^k for $k \geq 4$</i>	267
Samuel Irving Goldberg, <i>Totally geodesic hypersurfaces of Kaehler manifolds</i>	275
Donald Goldsmith, <i>On the multiplicative properties of arithmetic functions</i>	283
Jack D. Gray, <i>Local analytic extensions of the resolvent</i>	305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, <i>Commutativity theorems for nonassociative rings with a finite division ring homomorphic image</i>	325
André (Piotrowsky) De Korvin, <i>Normal expectations in von Neumann algebras</i>	333
James Donald Kuelbs, <i>A linear transformation theorem for analytic Feynman integrals</i>	339
W. Kuich, <i>Quasi-block-stochastic matrices</i>	353
Richard G. Levin, <i>On commutative, nonpotent archimedean semigroups</i>	365
James R. McLaughlin, <i>Functions represented by Rademacher series</i>	373
Calvin R. Putnam, <i>Singular integrals and positive kernels</i>	379
Harold G. Rutherford, II, <i>Characterizing primes in some noncommutative rings</i>	387
Benjamin L. Schwartz, <i>On interchange graphs</i>	393
Satish Shirali, <i>On the Jordan structure of complex Banach* algebras</i>	397
Earl J. Taft, <i>A counter-example to a fixed point conjecture</i>	405
J. Roger Teller, <i>On abelian pseudo lattice ordered groups</i>	411