FUNCTIONS REPRESENTED BY RADEMACHER SERIES

JAMES R. McLAUGHLIN
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A series of the form $\sum_{m=1}^{\infty} a_m r_m(t)$, where $\{a_m\}$ is a sequence of real numbers and $r_m(t)$ denotes the $m$th Rademacher function, sign $\sin(2^m \pi t)$, is called a Rademacher series (as usual, sign $0 = 0$).

Letting $f(t)$ denote the sum of this series whenever it exists, we shall investigate the effect that various conditions on $\{a_m\}$ have on the continuity, variation, and differentiability properties of $f$.

2. Continuity properties. We now prove

THEOREM (2.1). If $\sum |a_m| < \infty$, then $f(t)$ is continuous at dyadic irrationals (i.e., numbers not of the form $p/2^k$) and has right and left hand limits everywhere in $[0,1]$.

Proof. Under our hypothesis we have that $\sum a_m r_m(t)$ converges uniformly to $f(t)$, which implies our conclusion since the Rademacher functions are continuous at dyadic irrationals and have right and left hand limits everywhere in $[0,1]$.

In general, the right and left hand limits of $f(t)$ are unequal at dyadic rationals. We now investigate under what conditions we have equality and prove

THEOREM (2.2). If $\sum |a_m| < \infty$, then the following are equivalent:

(a) $a_k = \sum_{m=k+1}^{\infty} a_m$,
(b) $f(p2^{-k} + \varepsilon_n) \rightarrow f(p2^{-k})$ as $n \rightarrow \infty$,
(c) $f(p2^{-k} + \delta_n) \rightarrow f(p2^{-k})$ as $n \rightarrow \infty$,
(d) $f(p2^{-k} + \varepsilon_n) - f(p2^{-k} + \delta_n) \rightarrow 0$ as $n \rightarrow \infty$,

where $\{\varepsilon_n\}$ and $\{\delta_n\}$ are some positive and negative sequences tending to zero, and $p$ is an odd integer.

Proof. $f(p2^{-k} + t) - f(p2^{-k}) = \sum_{m=1}^{k-1} a_m r_m(p2^{-k} + t) - a_k r_k(t) + \sum_{m=k+1}^{\infty} a_m r_m(t) - \sum_{m=k+1}^{\infty} a_m r_m(p2^{-k})$,

since $r_m(p2^{-k} + t) = r_m(t)$ if $m \geq k + 1$, and $r_k(p2^{-k} + t) = -r_k(t)$. 373
Therefore,

\[ f(p2^{-k} + \varepsilon_n) - f(p2^{-k}) \to -a_k + \sum_{m=k+1}^{\infty} a_m \text{ as } n \to \infty. \]

This shows the equivalence of (a) and (b). A similar argument establishes the equivalence of (a), (c), and (d).

We have, at once, the following

**COROLLARY (2.1).** For absolutely convergent Rademacher series the following are equivalent:

(i) \( f(t) \) is continuous at \( p2^{-k} \) for some odd integer \( p \),
(ii) \( f(t) \) is continuous at \( p2^{-k} \) for all odd integers \( p \),
(iii) \( a_k = \sum_{m=k+1}^{\infty} a_m \).

**REMARKS.**

1. Notice that, if \( a_k = \sum_{m=k+1}^{\infty} a_m \) and \( a_{k+1} = \sum_{m=k+2}^{\infty} a_m \), then \( a_{k+1} = (a_k)/2 \).

2. Theorem (2.2) is false under the hypothesis that \( \sum |a_m| = \infty \) and \( a_m \to 0 \), since under these conditions we have that in every interval \( f(t) \) assumes every real number \( c \) times [2, p. 234, Th. 2].

This shows that the existence of the limit in the sense of Theorem (2.2) implies no relationship whatever between \( a_k \) and \( \sum_{m=k+1}^{\infty} a_m \). Also by choosing \( \{a_m\} \) such that \( \sum (a_m)^2 = \infty \) we see that the existence of the limit in the above sense does not even imply that \( \sum a_m r_m(t) \) converges in a set of positive measure [8, p. 212].

3. If \( f(t) = \sum a_m r_m(t) \) is essentially bounded, then \( \sum |a_m| < \infty \) (see [3]).

We now omit the condition that \( \sum |a_m| < \infty \) and prove

**THEOREM (2.3)** \( a_k = (a_{k-1})/2, k > 1, \) if either

\[
\lim_{n \to \infty} \left[ f(2^{-k} + p2^{-k+2} + \varepsilon_n) - f(2^{-k+1} + p2^{-k+2} + \varepsilon_n) \right] = \lim_{n \to \infty} \left[ f(2^{-k} + p2^{-k+2} + \delta_n) - f(2^{-k+1} + p2^{-k+2} + \delta_n) \right]
\]

(1)

or

\[
\lim_{n \to \infty} \left[ f(2^{-k+1} + p2^{-k+2} + \varepsilon_n) = f(3 \cdot 2^{-k} + p2^{-k+2} + \varepsilon_n) \right] = \lim_{n \to \infty} \left[ f(2^{-k+1} + p2^{-k+2} + \delta_n) - f(3 \cdot 2^{-k} + p2^{-k+2} + \delta_n) \right]
\]

(2)

where \( \varepsilon_n > 0, \delta_n < 0, \lim \varepsilon_n = \lim \delta_n = 0 \) and \( p \) is an integer.

**Proof.** If \( k > 1, \Delta(t) \)
\[ \equiv f(2^{-k} + p2^{-k+2} + t) - f(2^{-k+1} + p2^{-k+2} + t) \]
\[ = a_k[ r_k(2^{-k} + p2^{-k+2} + t) - r_{k-1}(2^{-k+1} + p2^{-k+2} + t)] + \cdots \]
\[ + a_{k-2}[ r_{k-2}(2^{-k} + p2^{-k+2} + t) - r_{k-3}(2^{-k+1} + p2^{-k+2} + t)] \]
\[ + a_{k-1}[ r_{k-1}(2^{-k} + t) + r_{k-1}(t)] + a_k[ -r_k(t) - r_k(t)] . \]

Thus,
\[ \lim_{n \to \infty} J(\varepsilon_n) = 2a_{k-1} - 2a_k \quad \text{and} \quad \lim_{n \to \infty} J(\delta_n) = 2a_k . \]

In view of (1) we have then \( 2a_k = a_{k-1} . \)
A similar proof will suffice if equation (2) is valid.

**Remark.** In much the same way we can prove a more general result, namely that if \( \{c_k\} \) has the property that
\[ \sum_{m=1}^{\infty} 1/ \prod_{k=1}^{m} (1 + c_k) = c^{-1} \neq 0 \]
is absolutely convergent, then
\[ f(t) = cf(0 + \sum_{m=1}^{\infty} r_m(t)/ \prod_{k=1}^{m} (1 + c_k) \]
if and only if for every \( k > 1 \) we have that in (1) the first limit equals \( c_k \) times the second.

We now utilize the concepts of approximate limits and approximately continuous functions (see [5, pp. 132, 219]). From Theorem (2.3), we deduce immediately.

**Corollary 2.2.** If the approximate limit of \( f(t) \) exists at either \( 2^{-k} + p2^{-k+2} \) and \( 2^{-k+1} + p2^{-k+3} \) or \( 2^{-k+1} + p2^{-k+2} \) and \( 3 \cdot 2^{-k} + p2^{-k+2} \) (where \( k > 1 \) and \( p \) is any integer), then \( a_k = (a_{k-1})/2 . \)

We now prove

**Corollary (2.3).** If \( F(t) \) is approximately continuous in \([0,1]\) and \( \sum a_m r_m(t) \) converges a.e. in \([0,1]\) to \( F(t) \), then
\[ F(t) = F(0) \cdot (1 - 2t), \quad a_m = F(0)/2^m (m = 1, 2, \cdots) . \]

**Proof.** Since \( F(t) \) is approximately continuous in \([0,1]\), we have that \( f(t) \) has approximate limits everywhere. Thus
\[ F(t) = C \sum r_m(t)/2^m \text{ a.e., } C \text{ being a constant.} \]

But, since \( \sum r_m(t)/2^m = 1 - 2t \) a.e. (see [7, p. 220]), this implies that
\[ F(t) = C(1 - 2t) \text{ a.e.} \]
which concludes our proof since $F(t)$ is approximately continuous.

REMARKS. 1. Corollary (2.2) shows that, if the approximate limits of $f(t)$ exist at certain dyadic rationals, then $a_m = C/2^m$ for $m \geq m_0$ (where $m_0, C$ are constants).

2. The conclusion of Corollary (2.3) was proved by Wang Si-Lei ([6, p. 704]; cf. [7, p. 221]) under the stronger hypothesis that $F(t)$ be continuous in $[0, 1]$. Wang's result can also be obtained from Theorem (2.2) and Remarks (1) and (3) following it.

3. Corollary (2.2) is a generalization of some theorems of Wang [6, Th. 1, 2, 3].

4. In Corollary (2.3), the condition "convergent a.e." cannot be replaced by "convergent in $E \subset [0, 1], |E| < 1" [6, p. 706].

3. Variational properties. A. I. Rubinstein has shown [4, p. 143] that if $\sum |a_m| 2^m < \infty$, then $f(t) \in \text{Lip}(1, 1)$.

In order to strengthen this result we now state the following lemma which follows from Minkowski's inequality:

**Lemma (3.1).** If $V_p(f_m)$ denotes the $p$th variation of $f_m(t)$, then

(i) if $0 < p \leq 1$, $V_p\left(\sum_{m=1}^{\infty} f_m\right) \leq \sum_{m=1}^{\infty} V_p(f_m)$;

(ii) if $p \geq 1$, $V_p\left(\sum_{m=1}^{\infty} f_m\right) \leq \sum_{m=1}^{\infty} V_p(f_m)$.

We will now prove

**Theorem (3.1).** (i) If $0 < p \leq 1$, then $\sum |a_m|^p 2^m < \infty$ implies $f(t)$ is of bounded $p$th variation;

(ii) if $p \geq 1$, then $\sum |a_m| 2^{m/p} < \infty$ implies $f(t)$ is of bounded $p$th variation;

(iii) if $0 < p \leq 1$, then $a_m \downarrow 0, \sum a_m^p 2^m = \infty$ implies

$$g(t) = \sum (-1)^m a_m r_m(t)$$

is not of bounded $p$th variation.

**Proof.** Parts (i) and (ii) are immediate by the lemma.

Also, setting $\{t_i\} = \{2^{-n-1} + i2^{-n}\}_{i=0}^{2^n-1}$ and $b_m = (-1)^m a_m$ we obtain

$$\sum_{i=1}^{2^n} |g(t_i) - g(t_{i-1})| = |2b_1 + \cdots + 2b_n|^p$$

$$+ 2 |2b_2 + \cdots + 2b_n|^p + \cdots + 2^{n-2} |-2b_{n-1} + 2b_n|^p$$

$$+ 2^{n-1} |2b_n|^p \geq \sum_{i=1}^{n} 2^{i-1} |2b_i|^p \to \infty \text{ as } n \to \infty .$$
This demonstrates Part (iii).

4. Differentiability properties. With regard to differentiability, L. A. Balasov has shown [1, p. 631] that $f(t)$ has a derivative at least one point if and only if

$$\lim_{m \to \infty} 2^m a_m = A \text{ exists}. \tag{3}$$

Balasov has demonstrated that this condition alone is not sufficient in order to have $f(t)$ differentiable a.e. [1, pp. 633-4]. He then proves that condition (3) and the relation

$$a_k \geq \sum_{m=k+1}^{\infty} a_m \text{ for every } k \geq 1$$

implies $f(t)$ is monotone in $[0,1]$, which of course implies differentiability almost everywhere.

We now prove

THEOREM (4.1). (i) If $\sum |a_m| 2^m < \infty$, then $f(t)$ is differentiable almost everywhere;

(ii) if $\{\varepsilon_m\}$ is any null sequence, then there exists a sequence $\{a_m\}$ satisfying

(a) $\sum |a_m 2^m \varepsilon_m| < \infty$,

(b) $f(t) = \sum a_m r_m(t)$ is differentiable nowhere.

Proof. Part (i) follows immediately from Theorem (3.1).

Part (ii). Since $\{\varepsilon_m\}$ is a null sequence, there exists an increasing sequence of positive integers $\{N_m\}$ such that

$$|\varepsilon_{N_m}| < 2^{-m}, \quad m = 1, 2, \ldots \tag{4}$$

Now set

$$a_m = 2^{-m}, \text{ if } m = N_i, \quad i = 2, 4, 6, \ldots$$

$$= 0, \text{ otherwise.}$$

Then (a) follows from condition (4), and (b) follows since Balasov’s condition (3) for differentiability is not satisfied.

REMARK. It would be interesting to know if the sum, $f(t)$, of a Rademacher series is of bounded variation whenever $f(t)$ is differentiable almost everywhere (as is the case for lacunary trigonometric series).
REFERENCES


Received June 27, 1967. This research was supported by a National Aeronautics and Space Administration Fellowship.

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Printed at Kokusai Bunken Insatsu sha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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