

Pacific Journal of Mathematics

A COUNTER-EXAMPLE TO A FIXED POINT CONJECTURE

EARL J. TAFT

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Let A be a finite-dimensional commutative Jordan algebra over a field F of characteristic zero. Then we may write $A = S + N$, S a semisimple subalgebra (Wedderburn factor), N the radical of A , [5], [6]. If G is a completely reducible group of automorphisms of A , then we may choose S to be invariant under G , [4]. If G is finite, then we showed in [10] that any two such G -invariant S were conjugate via an automorphism σ of A which centralizes G and which is a product of exponentials of nilpotent inner derivations of A of the form $\sum [R_{a_i}, R_{x_i}]$, x_i in N , a_i in A , where R_a is multiplication by a in A . It was conjectured in [10] that the various elements x_i and a_i which occur in the formulation of σ could be chosen as fixed points of G . This conjecture was based on analogous fixed point results proved for associative and Lie algebras, [7], [8], [9]. However, this conjecture is false, and we present in this note a simple counter-example.

We consider three-by-three matrices over F . Denoting by e_{ij} the usual matrix units, set $e = e_{11} + e_{22}$, $f = e_{33}$ and $x = e_{31}$. Consider the Jordan algebra A with basis e, f, x and multiplication table

	e	f	x
e	$2e$	0	x
f	0	$2f$	x
x	x	x	0

.

Clearly A has a one-dimensional radical $N = Fx$, and $S(0) = Fe + Ff$ is a Wedderburn factor of A . By [2], all Wedderburn factors are isomorphic, so are spanned by two orthogonal idempotents. The only idempotents (nonzero) of A are $(e/2) + \alpha x$, $(f/2) + \beta x$, α, β in F . The only pairs of orthogonal idempotents are $(e/2) + \alpha x$, $(f/2) - \alpha x$, α in F . Hence the Wedderburn factors of A are of the form $S(\alpha) = F(e + \alpha x) + F(f - \alpha x)$, and clearly $\alpha \rightarrow S(\alpha)$ is one-to-one.

A has two types of automorphisms, as can be seen by a direct check. The first type $A(\delta, \pi)$, δ, π in F , $\pi \neq 0$, is given by:

$$A(\delta, \pi) \begin{cases} e \rightarrow f + \delta x \\ f \rightarrow e - \delta x \\ x \rightarrow \pi x \end{cases}$$

The second type $B(\delta, \pi)$, δ, π in F , $\pi \neq 0$, is given by:

$$B(\delta, \pi) \begin{cases} e \rightarrow e + \delta x \\ f \rightarrow f - \delta x \\ x \rightarrow \pi x \end{cases}$$

A calculation shows that $S(\alpha) B(\delta, \pi) = S(\alpha\pi + \delta)$, so that if $\pi \neq 1$, $S((1 - \pi)^{-1}\delta)$ is the only $B(\delta, \pi)$ -invariant Wedderburn factor of A . If $\delta \neq 0$, then $B(\delta, 1)$ fixes no Wedderburn factor, and $B(0, 1) = I$, the identity mapping of A .

Turning to $A(\delta, \pi)$, we have that $S(\alpha)A(\delta, \pi) = S(-\alpha\pi - \delta)$. Hence if $\pi \neq -1$, $S(-\delta(1 + \pi)^{-1})$ is the only $A(\delta, \pi)$ -invariant Wedderburn factor of A . If $\delta \neq 0$, then $A(\delta, -1)$ fixes no Wedderburn factor, but $A(0, -1)$ fixes all Wedderburn factors $S(\alpha)$. Let G be the group of order two generated by $A(0, -1)$:

$$A(0, -1) \begin{cases} e \rightarrow f \\ f \rightarrow e \\ x \rightarrow -x \end{cases}$$

Note that $e - f$ and x are eigenvectors for the eigenvalue -1 of $A(0, -1)$, so that $F(e + f)$ is the fixed point space of G . $R_{e+f} = 2I$, and N has no nonzero fixed points under G , which disproves the conjecture.

In checking the result of [10] in this example, let $D = [R_{e-f}, R_x] = R_{e-f}R_x - R_xR_{e-f}$. Then one can check that

$$\sigma = \exp\left(\left(\frac{\beta - \alpha}{2}\right)D\right) = I + \frac{\beta - \alpha}{2}D$$

will map $S(\alpha)$ onto $S(\beta)$ for any α, β in F . Since $e - f$ and x are in the -1 -eigenspace of $A(0, -1)$, the rule $g^{-1}R_ag = R_{ag}$ for a in A , g an automorphism of A , shows that D commutes with $A(0, -1)$, so that σ centralizes G . This leads to the more complicated conjecture that one can formulate σ in terms of inner derivations $[R_a, R_x]$, a in A , x in N , such that for any g in G , a and x are eigenvectors of g corresponding to eigenvalues $\alpha(g)$ and $\beta(g)$ respectively, such that $\alpha(g)\beta(g) = 1$. Such a σ will centralize G . We also note that this conjecture and the fixed point conjecture are still open for alternative algebras (see [10] for a precise formulation), although the fixed point conjecture now seems unlikely for alternative algebras, in view of the

above counter-example for Jordan algebras, due to the close relation between alternative and Jordan algebras, [3]. We also remark that for completely reducible G , the existence of a σ centralizing G is still an open question. If $N^2 = 0$, this is trivial (see [10], §5), and the difficulty lies in the case $N^2 \neq 0$. We also note that if F is any field of characteristic not two, then our example has A/N separable and $N^2 = 0$, in which case the Wedderburn-Malcev properties hold, [1], [2], [6], and any finite group G of order not divisible by the characteristic of F will fix a Wedderburn factor, [6]. So our example also shows that the fixed point conjecture is false for the case $N^2 = 0$, R/N separable.

We conclude with an example of an infinite group G which illustrates the conjecture for completely reducible G that σ can be chosen to centralize G , in a case where $N^2 \neq 0$. Again considering three-by-three matrices over F , let $e = e_{11} + e_{33}$, $x = e_{12}$, $y = e_{23}$, $z = e_{13}$. Let A be the Jordan algebra with basis e, x, y, z and multiplication table

	e	x	y	z
e	$2e$	x	y	$2z$
x	x	0	z	0
y	y	z	0	0
z	$2z$	0	0	0

Clearly the radical N of A is $N = Fx + Fy + Fz$, $N^2 = Kz$ and $N^3 = 0$. Clearly $S(0, 0) = Ke$ is a Wedderburn factor, and if we calculate the elements f for which $f^2 = 2f$, we find

$$f = e + \alpha x + \beta y - \alpha\beta z, \alpha, \beta \in F.$$

Since all Wedderburn factors are isomorphic (we are assuming characteristic zero), the Wedderburn factors are of the form

$$S(\alpha, \beta) = F(e + \alpha x + \beta y - \alpha\beta z),$$

and the correspondence $(\alpha, \beta) \rightarrow S(\alpha, \beta)$ is one-to-one on $F \times F$.

Let $\delta \in F$, $\phi \in F$, $\phi \neq 0, 1$. Let $A(\delta, \phi)$ be the automorphism of A given by:

$$A(\delta, \phi) \begin{cases} e \rightarrow e + \delta y \\ x \rightarrow x - \delta z \\ y \rightarrow \phi y \\ z \rightarrow \phi z \end{cases}.$$

$A(\delta, \phi)$ is completely reducible, since A has a basis of eigenvectors $y, z, (1 - \phi)e + \delta y, (1 - \phi)x - \delta z$, the latter two being fixed points of $A(\delta, \phi)$. One can check that $S(\alpha, \beta)A(\delta, \phi) = S(\alpha, \delta + \beta\phi)$, so that $S(\alpha, \delta(1 - \phi)^{-1})$ is fixed by G , the group generated by $A(\delta, \phi)$, for any α in F . For α, α' in F , set

$$D = (\alpha' - \alpha)(1 - \phi)^{-2}[R_{(1-\phi)e+\delta y}, R_{(1-\phi)x-\delta z}].$$

Then one can calculate that $\sigma = \exp D = I + D + (D^2/2)$ carries $S(\alpha, \delta(1 - \phi)^{-1})$ onto $S(\alpha', \delta(1 - \phi)^{-1})$, and centralizes G since the elements $(1 - \phi)e + \delta y, (1 - \phi)x - \delta z$ are fixed points of $A(\delta, \phi)$. Note that if ϕ is not a root of unity, then G is an infinite group.

Another automorphism $B(\delta, \tau)$ of A , for δ, τ in $F, \tau \neq 0$, is given by:

$$B(\delta, \tau) \begin{cases} e \rightarrow e - \delta\tau x + \delta y + \delta^2\tau z \\ x \rightarrow \tau^{-1}y + \delta z \\ y \rightarrow \tau x - \delta\tau z \\ z \rightarrow z \end{cases}.$$

$B(\delta, \tau)$ has a three-dimensional fixed point space spanned by $e + \delta y, z$ and $\tau x + y$, and an eigenvector $\tau x - y - \delta\tau z$ for the eigenvalue -1 , so that $B(\delta, \tau)$ is completely reducible. Actually $B(\delta, \tau)^2 = I$, so G here is a group of order two. One calculates that $S(\alpha, \beta)B(\delta, \tau) = S(-\delta\tau + \beta\tau, \delta + \alpha\tau^{-1})$. Hence $S(\alpha, \delta + \alpha\tau^{-1})$ is G -invariant for any $\alpha \in F$. Set $D' = \tau^{-1}(\alpha' - \alpha)[R_{e+\delta y}, R_{\tau x+y}]$ for $\alpha, \alpha' \in F$. Then

$$\sigma = \exp D' = I + D' + \frac{(D')^2}{2}$$

carries $S(\alpha, \delta + \alpha\tau^{-1})$ onto $S(\alpha', \delta + \alpha'\tau^{-1})$, and centralizes G since $e + \delta y$ and $\tau x + y$ are fixed points of $B(\delta, \tau)$. Hence, in this case, the fixed point property holds, although, as we have seen in our first example, it does not hold for every finite group G .

REFERENCES

1. B. Harris, *Derivations of Jordan algebras*, Pacific J. Math. **9** (1959), 495-512.
2. N. Jacobson, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc. **70** (1951), 509-530.
3. ———, *Structure of alternative and Jordan bimodules*, Osaka Math. J. **6** (1954), 1-71.
4. G. D. Mostow, *Fully reducible subgroups of algebraic groups*, Amer. J. Math. **78** (1956), 200-221.
5. A. J. Penico, *The Wedderburn principal theorem for Jordan algebras*, Trans. Amer. Math. Soc. **70** (1951), 404-421.
6. E. J. Taft, *Invariant Wedderburn factors*, Illinois J. Math. **1** (1957), 565-573.
7. ———, *Uniqueness of invariant Wedderburn factors*, Illinois J. Math. **6** (1962),

353-356.

8. ———, *Invariant Levi factors*, Michigan Math. J. **9** (1962), 65-68.
9. ———, *Orthogonal conjugacies in associative and Lie algebras*, Trans. Amer. Math. Soc. **113** (1964), 18-29.
10. ———, *Invariant splitting in Jordan and alternative algebras*, Pacific J. Math. **15** (1965), 1421-1427.

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Vol. 27, No. 2

February, 1968

Leonard E. Baum and George Roger Sell, <i>Growth transformations for functions on manifolds</i>	211
Henry Gilbert Bray, <i>A note on CLT groups</i>	229
Paul Robert Chernoff, Richard Anthony Rasala and William Charles Waterhouse, <i>The Stone-Weierstrass theorem for valuable fields</i>	233
Douglas Napier Clark, <i>On matrices associated with generalized interpolation problems</i>	241
Richard Brian Darst and Euline Irwin Green, <i>On a Radon-Nikodym theorem for finitely additive set functions</i>	255
Carl Louis DeVito, <i>A note on Eberlein's theorem</i>	261
P. H. Doyle, III and John Gilbert Hocking, <i>Proving that wild cells exist</i>	265
Leslie C. Glaser, <i>Uncountably many almost polyhedral wild $(k - 2)$-cells in E^k for $k \geq 4$</i>	267
Samuel Irving Goldberg, <i>Totally geodesic hypersurfaces of Kaehler manifolds</i>	275
Donald Goldsmith, <i>On the multiplicative properties of arithmetic functions</i>	283
Jack D. Gray, <i>Local analytic extensions of the resolvent</i>	305
Eugene Carlyle Johnsen, David Lewis Outcalt and Adil Mohamed Yaqub, <i>Commutativity theorems for nonassociative rings with a finite division ring homomorphic image</i>	325
André (Piotrowsky) De Korvin, <i>Normal expectations in von Neumann algebras</i>	333
James Donald Kuelbs, <i>A linear transformation theorem for analytic Feynman integrals</i>	339
W. Kuich, <i>Quasi-block-stochastic matrices</i>	353
Richard G. Levin, <i>On commutative, nonpotent archimedean semigroups</i>	365
James R. McLaughlin, <i>Functions represented by Rademacher series</i>	373
Calvin R. Putnam, <i>Singular integrals and positive kernels</i>	379
Harold G. Rutherford, II, <i>Characterizing primes in some noncommutative rings</i>	387
Benjamin L. Schwartz, <i>On interchange graphs</i>	393
Satish Shirali, <i>On the Jordan structure of complex Banach* algebras</i>	397
Earl J. Taft, <i>A counter-example to a fixed point conjecture</i>	405
J. Roger Teller, <i>On abelian pseudo lattice ordered groups</i>	411