A COUNTER-EXAMPLE TO A FIXED POINT CONJECTURE

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Let $A$ be a finite-dimentional commutative Jordan algebra over a field $F$ of characteristic zero. Then we may write $A = S + N$, $S$ a semisimple subalgebra (Wedderburn factor), $N$ the radical of $A$, [5], [6]. If $G$ is a completely reducible group of automorphisms of $A$, then we may choose $S$ to be invariant under $G$, [4]. If $G$ is finite, then we showed in [10] that any two such $G$-invariant $S$ were conjugate via an automorphism $\sigma$ of $A$ which centralizes $G$ and which is a product of exponentials of nilpotent inner derivations of $A$ of the form $\sum [R_{xi}, R_{zi}]$, $x_i$ in $N$, $a_i$ in $A$, where $R_a$ is multiplication by $a$ in $A$. It was conjectured in [10] that the various elements $x_i$ and $a_i$ which occur in the formulation of $\sigma$ could be chosen as fixed points of $G$. This conjecture was based on analogous fixed point results proved for associative and Lie algebras, [7], [8], [9]. However, this conjecture is false, and we present in this note a simple counter-example.

We consider three-by-three matrices over $F$. Denoting by $e_{ij}$ the usual matrix units, set $e = e_{11} + e_{22}$, $f = e_{33}$ and $x = e_{31}$. Consider the Jordan algebra $A$ with basis $e, f, x$ and multiplication table

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
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<tbody>
<tr>
<td>$e$</td>
<td>$2e$</td>
<td>0</td>
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<tr>
<td>$f$</td>
<td>0</td>
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<td>$x$</td>
<td>$x$</td>
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<td>0</td>
</tr>
</tbody>
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Clearly $A$ has a one-dimensional radical $N = Fx$, and $S(0) = Fe + Ff$ is a Wedderburn factor of $A$. By [2], all Wedderburn factors are isomorphic, so are spanned by two orthogonal idempotents. The only idempotents (nonzero) of $A$ are $(e/2) + \alpha x$, $(f/2) + \beta x$, $\alpha, \beta$ in $F$. The only pairs of orthogonal idempotents are $(e/2) + \alpha x$, $(f/2) - \alpha x$, $\alpha$ in $F$. Hence the Wedderburn factors of $A$ are of the form $S(\alpha) = F(e + \alpha x) + F(f - \alpha x)$, and clearly $\alpha \rightarrow S(\alpha)$ is one-to-one.

A has two types of automorphisms, as can be seen by a direct check. The first type $A(\delta, \pi)$, $\delta, \pi$ in $F$, $\pi \neq 0$, is given by:
The second type $B(\delta, \pi)$, $\delta, \pi$ in $F$, $\pi \neq 0$, is given by:

$$
\begin{align*}
A(\delta, \pi) & \begin{cases}
eq f + \delta x \\
f \rightarrow e - \delta x \\
x \rightarrow \pi x
\end{cases}
, \\
B(\delta, \pi) & \begin{cases}
eq e + \delta x \\
f \rightarrow f - \delta x \\
x \rightarrow \pi x
\end{cases}
\end{align*}
$$

A calculation shows that $S(\alpha) B(\delta, \pi) = S(\alpha \pi + \delta)$, so that if $\pi \neq 1$, $S((1 - \pi)^{-1} \delta)$ is the only $B(\delta, \pi)$-invariant Wedderburn factor of $A$. If $\delta \neq 0$, then $B(\delta, 1)$ fixes no Wedderburn factor, and $B(0, 1) = I$, the identity mapping of $A$.

Turning to $A(\delta, \pi)$, we have that $S(\alpha) A(\delta, \pi) = S(-\alpha \pi + \delta)$. Hence if $\pi \neq -1$, $S(-\delta(1 + \pi)^{-1})$ is the only $A(\delta, \pi)$-invariant Wedderburn factor of $A$. If $\delta \neq 0$, then $A(\delta, -1)$ fixes no Wedderburn factor, but $A(0, -1)$ fixes all Wedderburn factors $S(\alpha)$. Let $G$ be the group of order two generated by $A(0, -1)$:

$$
\begin{align*}
A(0, -1) & \begin{cases}
eq f \\
f \rightarrow e \\
x \rightarrow -x
\end{cases}
\end{align*}
$$

Note that $e - f$ and $x$ are eigenvectors for the eigenvalue $-1$ of $A(0, -1)$, so that $F(e + f)$ is the fixed point space of $G$. $R_{e+f} = 2I$, and $N$ has no nonzero fixed points under $G$, which disproves the conjecture.

In checking the result of [10] in this example, let $D = [R_{e-f}, R_x] = R_{e-f}R_x - R_xR_{e-f}$. Then one can check that

$$
\sigma = \exp \left( \frac{\beta - \alpha}{2} D \right) = I + \frac{\beta - \alpha}{2} D
$$

will map $S(\alpha)$ onto $S(\beta)$ for any $\alpha, \beta$ in $F$. Since $e - f$ and $x$ are in the $-1 -$ eigenspace of $A(0, -1)$, the rule $g^{-1}R_{e}g = R_{ag}$ for $a$ in $A$, $g$ an automorphism of $A$, shows that $D$ commutes with $A(0, -1)$, so that $\sigma$ centralizes $G$. This leads to the more complicated conjecture that one can formulate $\sigma$ in terms of inner derivations $[R_a, R_x]$, $a$ in $A$, $x$ in $N$, such that for any $g$ in $G$, $a$ and $x$ are eigenvectors of $g$ corresponding to eigenvalues $\alpha(g)$ and $\beta(g)$ respectively, such that $\alpha(g) \beta(g) = 1$. Such a $\sigma$ will centralize $G$. We also note that this conjecture and the fixed point conjecture are still open for alternative algebras (see [10] for a precise formulation), although the fixed point conjecture now seems unlikely for alternative algebras, in view of the
above counter-example for Jordan algebras, due to the close relation between alternative and Jordan algebras, [3]. We also remark that for completely reducible \( G \), the existence of a \( \sigma \) centralizing \( G \) is still an open question. If \( N^2 = 0 \), this is trivial (see [10], §5), and the difficulty lies in the case \( N^2 \neq 0 \). We also note that if \( F \) is any field of characteristic not two, then our example has \( A/N \) separable and \( N^2 = 0 \), in which case the Wedderburn-Malcev properties hold, [1], [2], [6], and any finite group \( G \) of order not divisible by the characteristic of \( F \) will fix a Wedderburn factor, [6]. So our example also shows that the fixed point conjecture is false for the case \( N^2 = 0, \ R/N \) separable.

We conclude with an example of an infinite group \( G \) which illustrates the conjecture for completely reducible \( G \) that \( \sigma \) can be chosen to centralize \( G \), in a case where \( N^2 \neq 0 \). Again considering three-by-three matrices over \( F \), let \( e = e_{11} + e_{33}, \ x = e_{12}, \ y = e_{33}, \ z = e_{13}. \) Let \( A \) be the Jordan algebra with basis \( e, x, y, z \) and multiplication table

\[
\begin{array}{cccc}
e & x & y & z \\
e & 2e & x & y & 2z \\
x & x & 0 & z & 0 \\
y & y & z & 0 & 0 \\
z & 2z & 0 & 0 & 0 \\
\end{array}
\]

Clearly the radical \( N \) of \( A \) is \( N = Fx + Fy + Fz, N^2 = Kz \) and \( N^3 = 0 \). Clearly \( S(0, 0) = Ke \) is a Wedderburn factor, and if we calculate the elements \( f \) for which \( f^2 = 2f \), we find

\[
f = e + \alpha x + \beta y - \alpha \beta z, \ \alpha, \ \beta \in F.
\]

Since all Wedderburn factors are isomorphic (we are assuming characteristic zero), the Wedderburn factors are of the form

\[
S(\alpha, \beta) = F(e + \alpha x + \beta y - \alpha \beta z),
\]

and the correspondence \((\alpha, \beta) \rightarrow S(\alpha, \beta)\) is one-to-one on \( F \times F \).

Let \( \delta \in F, \ \phi \in F, \ \phi \neq 0, 1 \). Let \( A(\delta, \phi) \) be the automorphism of \( A \) given by:

\[
A(\delta, \phi) \left\{ \begin{array}{l}
e \rightarrow e + \delta y \\
x \rightarrow x - \delta z \\
y \rightarrow \phi y \\
z \rightarrow \phi z \\
\end{array} \right.
\]
$A(\delta, \phi)$ is completely reducible, since $A$ has a basis of eigenvectors $y, z, (1 - \phi)e + \delta y, (1 - \phi)x - \delta z$, the latter two being fixed points of $A(\delta, \phi)$. One can check that $S(\alpha, \beta)A(\delta, \phi) = S(\alpha, \delta + \beta \phi)$, so that $S(\alpha, \delta(1 - \phi)^{-1})$ is fixed by $G$, the group generated by $A(\delta, \phi)$, for any $\alpha$ in $F$. For $\alpha, \alpha'$ in $F$, set

$$D = (\alpha' - \alpha)(1 - \phi)^{-2}[R_{(1 - \phi)e + \delta y}, R_{(1 - \phi)x - \delta z}].$$

Then one can calculate that $\sigma = \exp D = I + D + (D^2/2)$ carries $S(\alpha, \delta(1 - \phi)^{-1})$ onto $S(\alpha', \delta(1 - \phi)^{-1})$, and centralizes $G$ since the elements $(1 - \phi)e + \delta y, (1 - \phi)x - \delta z$ are fixed points of $A(\delta, \phi)$. Note that if $\phi$ is not a root of unity, then $G$ is an infinite group.

Another automorphism $B(\delta, \tau)$ of $A$, for $\delta, \tau$ in $F, \tau \neq 0$, is given by:

$$B(\delta, \tau)\begin{cases}
eq e \rightarrow e - \delta \tau x + \delta y + \delta^2 \tau z \\
x \rightarrow \tau^{-1}y + \delta z \\
y \rightarrow \tau x - \delta \tau z \\
z \rightarrow z
\end{cases}.$$  

$B(\delta, \tau)$ has a three-dimensional fixed point space spanned by $e + \delta y, z$ and $\tau x + y$, and an eigenvector $\tau x - y - \delta \tau z$ for the eigenvalue $-1$, so that $B(\delta, \tau)$ is completely reducible. Actually $B(\delta, \tau)^2 = I$, so $G$ here is a group of order two. One calculates that $S(\alpha, \beta)B(\delta, \tau) = S(-\delta \tau + \beta \tau, \delta + \alpha \tau^{-1})$. Hence $S(\alpha, \delta + \alpha \tau^{-1})$ is $G$-invariant for any $\alpha \in F$. Set $D' = \tau^{-1}(\alpha' - \alpha)[R_{e + \delta y}, R_{\tau x + y}]$ for $\alpha, \alpha' \in F$. Then

$$\sigma = \exp D' = I + D' + \frac{(D')^2}{2}$$

carries $S(\alpha, \delta + \alpha \tau^{-1})$ onto $S(\alpha', \delta + \alpha' \tau^{-1})$, and centralizes $G$ since $e + \delta y$ and $\tau x + y$ are fixed points of $B(\delta, \tau)$. Hence, in this case, the fixed point property holds, although, as we have seen in our first example, it does not hold for every finite group $G$.

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Received August 2, 1967. Research supported by National Science Foundation Grant GP-7162.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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