ON \((J, M, m)\)-EXTENSIONS OF ORDER SUMS OF DISTRIBUTIVE LATTICES

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In the first section of this paper a characterization of the order sum of a family \(\{L_a\}_{a \in S}\) of distributive lattices is given which is analogous to the characterization of a free distributive lattice as one generated by an independent set. We then consider the collection \(Q\) of order sums obtained by taking different partial orderings on \(S\). A natural partial ordering is defined on \(Q\) and its maximal and minimal elements are characterized.

Let \(J\) and \(M\) be collections of nonempty subsets of a distributive lattice \(L\), and \(m\) a cardinal. We define a \((J, M, m)\)-extension \((\psi, E)\) of \(L\), where \(E\) is a \(m\)-complete distributive lattice and \(\psi: L \to E\) is a \((J, M)\)-monomorphism. In the last section we define a \(m\)-order sum of a family of distributive lattices \(\{L_a\}_{a \in S}\). The main result here is that the \(m\)-order sum exists if the order sum \(L\) of \(\{L_a\}_{a \in S}\) has a \((J, M, m)\)-extension, where \(J\) and \(M\) are certain collections of subsets of \(L\). These results are analogous to R. Sikorski's work in Boolean algebras (e.g., [6]).

1. Order sums. Let \(S\) be a fixed set and \(\{L_a\}_{a \in S}\) a fixed collection of distributive lattices. From [2] it follows that for each poset \(P = (S, \leq)\), there exists a pair \((\{\varphi_a\}_{a \in S}, L(P))\), where \(L(P)\) is a distributive lattice, and for each \(a \in S\), \(\varphi_a: L_a \to L(P)\) is a monomorphism such that:

\[(1.1)\] \(L\) is generated by \(\bigcup_{a \in S} \varphi_a(L_a)\).

\[(1.2)\] If \(a < b\) then \(\varphi_a(x) < \varphi_b(y)\), for all \(x \in L_a\) and \(y \in L_b\).

\[(1.3)\] If \(M\) is a distributive lattice and \(\{f_a: L_a \to M\}_{a \in S}\) is a family of homomorphisms such that \(f_a(x) \leq f_b(y)\) whenever \(a < b\), \(x \in L_a\) and \(y \in L_b\), then there exists a homomorphism \(f: L(P) \to M\) such that \(f \varphi_a = f_a\) for each \(a \in S\).

The pair \((\{\varphi_a\}_{a \in S}, L(P))\) will be called an order sum of \(\{L_a\}_{a \in S}\) over \(P\).

Let \(P\) be the family of all posets of the form \((S, \leq)\) and let \(Q = \{(\{\varphi_a\}_{a \in S}, L(P)) \mid P \in P\}\). For \((\{\varphi_a\}_{a \in S}, L(P))\) and \((\{\theta_a\}_{a \in S}, L(P'))\) in \(Q\) we write

\[(1.4)\] \((\{\varphi_a\}_{a \in S}, L(P)) \leq (\{\theta_a\}_{a \in S}, L(P'))\) provided:

\[(1.5)\] there is a homomorphism \(f: L(P') \to L(P)\) such that \(f \theta_a = \varphi_a\) for each \(a \in S\).

Note that \((1.5)\) implies \(f\) is an epimorphism. If \(f\) is an isomor-
phism, we say that \((\{\varphi_a\}_{a \in S}, L(P))\) is isomorphic with \((\{\varphi_a\}_{a \in S}, L(P'))\). Isomorphism in this sense is an equivalence relation \(\simeq\), and [2, Th. 1.2] implies that any two order sums over \(P\) are isomorphic. By identifying isomorphs, (1.4) determines a partial ordering on the equivalence classes of \(Q/\simeq\).

**Definition 1.1.** Suppose \(P \in P\) and \(\{N_a\}_{a \in S}\) is a family of sublattices of a distributive lattice \(N\). The family \(\{N_a\}_{a \in S}\) is called \(P\)-independent if whenever \(\alpha_1, \ldots, \alpha_m\) are distinct elements of \(S\), \(\alpha_{m+1}, \ldots, \alpha_n\) are distinct elements of \(S\) and \(x_i \in N_{\alpha_i}\) for \(i = 1, \ldots, n\) then

\[
(1.6) \quad x_1 \cdot \cdots \cdot x_m \leq x_{m+1} + \cdots + x_n \text{ if and only if } \quad (1.7) \text{ for some } i \text{ and } j, \text{ either } \alpha_i < \alpha_j \text{ or } \alpha_i = \alpha_j \text{ and } x_i \leq x_j, \text{ where } 1 \leq i \leq m \text{ and } m + 1 \leq j \leq n.
\]

**Lemma 1.2.** Suppose \(N\) and \(M\) are distributive lattices and \(\{N_a\}_{a \in S}\) is a collection of sublattices of \(N\) such that \(\bigcup_{a \in S} N_a\) generates \(N\). A necessary and sufficient condition for a family \(\{f_a: N_a \to M\}_{a \in S}\) of homomorphisms to have a common extension on \(N\) is that if \(\alpha_1, \ldots, \alpha_m\) are distinct members of \(S\), \(\alpha_{m+1}, \ldots, \alpha_n\) are distinct members of \(S\), \(x_i \in N_{\alpha_i}\) for \(i = 1, \ldots, n\) and

\[
(1.8) \quad x_1 \cdot \cdots \cdot x_m \leq x_{m+1} + \cdots + x_n \text{ then } \quad (1.9) \quad f_{a_1}(x_1) \cdot \cdots \cdot f_{a_m}(x_m) \leq f_{a_{m+1}}(x_{m+1}) + \cdots + f_{a_n}(x_n).
\]

**Proof.** The necessity is clear. Now if \(x \in N_\alpha \cap N_\beta\) then by (1.9), \(x \leq x\) implies that \(f_\alpha(x) = f_\beta(x)\). So the function \(f: \bigcup_{a \in S} N_a \to M\) defined by \(f(x) = f_a(x)\) if \(x \in L_\alpha\) makes sense and has the property that if \(A\) and \(B\) are finite nonempty subsets of \(\bigcup_{a \in S} N_a\), then \(\Pi_N(A) \leq \Sigma_N(B)\) implies \(\Pi_M(f(A)) \leq \Sigma_M(f(B))\). By [1, Lemma 1.7], \(f\) can be extended to a homomorphism \(f': N \to M\). This is the required extension.

**Theorem 1.3.** The pair \((\{\theta_a\}_{a \in S}, L)\) is the order sum of \(\{L_a\}_{a \in S}\) over \(P \in P\) if and only if \(\{\theta_a: L_a \to L\}_{a \in S}\) is a family of monomorphisms such that:

\[
(1.10) \quad \bigcup_{a \in S} \theta_a(L_a) \text{ generates } L, \text{ and } \quad (1.11) \quad \{\theta_a(L_a)\}_{a \in S} \text{ is } P\text{-independent.}
\]

**Proof.** For the sufficiency suppose first that \(\alpha < \beta\). By (1.11) \(\theta_\alpha(x) \leq \theta_\beta(y)\) for all \(x \in L_\alpha\), \(y \in L_\beta\). But if \(\theta_\beta(y) \leq \theta_\alpha(x)\) then \(\beta \leq \alpha\). Hence (1.2) is satisfied. Now assume the hypothesis of (1.3). It is sufficient to show that the family \(\{f_a \theta_a^{-1}: \theta_a(L_a) \to M\}_{a \in S}\) has a common extension on \(L\). So if

\[
\theta_a(x_1) \cdot \cdots \cdot \theta_a(x_m) \leq \theta_{a_{m+1}}(x_{m+1}) + \cdots + \theta_a(x_n)
\]
where $\alpha_1, \ldots, \alpha_m$ are distinct and $\alpha_{m+1}, \ldots, \alpha_n$ are distinct then by (1.11) there exists $p, q$ such that $\alpha_p < \alpha_q$ or $\alpha_p = \alpha_q$ and $\theta_{\alpha_p}(x_p) \leq \theta_{\alpha_q}(x_q)$, where $1 \leq p \leq m$ and $m + 1 \leq q \leq n$. In any case $f_{\alpha_p}(x_p) \leq f_{\alpha_q}(x_q)$ and so

$$\Pi_{i=1}^n f_{\alpha_i}^{-1} \theta_{\alpha_i}(x_i) \leq \Sigma_{j=m+1}^n f_{\alpha_j}^{-1} \theta_{\alpha_j}(x_j).$$

The result now follows from Lemma 1.2. The converse is essentially [2, Th. 1.9].

The set $P$ can be partially ordered as follows. If $P, P' \in P$ then $P \preceq P'$ provided $P \subseteq P'$, as sets of ordered pairs. It is immediate that $P$ has a greatest element—the trivial partial ordering on $S$. Also, it can be shown that $P$ is minimal in $P$ if and only if $P$ is a chain.

**Theorem 1.4.** $P \cong Q/\sim$.

*Proof.* It is sufficient to show that for $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$, $(\{\theta_\alpha\}_{\alpha \in S}, L(P')) \in Q$:

(1.12) $P \preceq P'$

if and only if

(1.13) $(\{\varphi_\alpha\}_{\alpha \in S}, L(P)) \preceq (\{\theta_\alpha\}_{\alpha \in S}, L(P'))$.

If $P \preceq P'$, then $\{\varphi_\alpha : L_\alpha \rightarrow L(P)\}_{\alpha \in S}$ is a family of homomorphisms with the property that if $\alpha < \beta$ (in $P'$) then $\varphi_\alpha(x) < \varphi_\beta(y)$ for all $x \in L_\alpha$, $y \in L_\beta$. So by (1.3), we have (1.13). Conversely, suppose (1.5) holds and $\alpha < \beta$ (in $P'$). Letting $x \in L_\alpha$ and $y \in L_\beta$, we have $\theta_\alpha(x) < \theta_\beta(y)$ so $\varphi_\alpha(x) = f_\theta_\alpha(x) \leq f_\theta_\beta(y) = \varphi_\beta(y)$. Since $\{\varphi_\alpha(L_\alpha)\}_{\alpha \in S}$ is $P$-independent, $\alpha \leq \beta$ (in $P$). It follows that $P' \preceq P$.

**Corollary 1.5.** $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))/\sim$ is the greatest element in $Q/\sim$ if and only if $L(P)$ is the free product of $\{L_\alpha\}_{\alpha \in S}$. Furthermore, $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))/\sim$ is minimal in $Q/\sim$ if and only if $L(P)$ is an ordinal sum of $\{L_\alpha\}_{\alpha \in S}$.

*Proof.* The definitions of free product and ordinal sum can be found in [7, §9] and [2, Definition 1.3]. The result then follows from Theorem 1.4 and the remark following Theorem 1.3.

For the remainder of this section, let $(\{\varphi_\alpha\}_{\alpha \in S}, L(P))$ be a fixed member of $Q$.

A lattice $L$ is said to be *conditionally impliciative* if for each pair $x, y \in L$ such that $x \not\leq y$ there is an element $x \rightarrow y$ with the property that $x \cdot z \leq y$ if and only if $z \leq x \rightarrow y$. Note that conditionally
implicative lattices are distributive. The following theorem, which we stated without proof in [2], is the converse of [2, Th. 2.5].

**Theorem 1.6.** If \( L(P) \) is conditionally implicative then \( L_a \) is conditionally implicative for each \( \alpha \in S \).

**Proof.** Let \( x, y \in L_a \) and \( x \leq y \). Then \( \varphi_a(x) \rightarrow \varphi_a(y) \) exists in \( L(P) \) and equals a sum of \( m \) products, each of the form

\[
\varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n).
\]

We can assume \( \gamma_i \leq \gamma_j \) for \( i \neq j \). Now

\[
\varphi_a(x)(\varphi_{i_1}(x_1) \cdots \varphi_{i_n}(x_n)) \leq \varphi_a(x)(\varphi_a(x) \rightarrow \varphi_a(y)) \leq \varphi_a(y).
\]

By (1.11) there exists \( p \) such that \( \gamma_p < \alpha \) or \( \gamma_p = \alpha \) and \( xx_p \leq y \). But in any case \( \varphi_a(x) \varphi_{i_p}(x_p) \leq \varphi_a(y) \). Hence

(1.14) \( \varphi_{i_p}(x_p) \leq \varphi_a(x) \rightarrow \varphi_a(y) \).

Choosing an element \( \varphi_{i_j}(y_j) \), that satisfies (1.14), from each of the \( m \) summands of \( \varphi_a(x) \rightarrow \varphi_a(y) \), we have:

\[
\sum_{j=1}^m \varphi_{i_j}(y_j) \leq \varphi_a(x) \rightarrow \varphi_a(y) \leq \sum_{j=1}^m \varphi_{i_j}(y_j),
\]

and so \( \varphi_a(x) \rightarrow \varphi_a(y) = \sum_{j=1}^m \varphi_{i_j}(y_j) \), where \( \beta_i \leq \beta_j \) for \( i \neq j \). For each \( j \), \( \varphi_a(x) \varphi_{i_j}(y_j) \leq \varphi_a(x)(\varphi_a(x) \rightarrow \varphi_a(y)) \leq \varphi_a(y) \), and since \( x \leq y \), we have:

\[
\beta_j \leq \alpha \quad \text{for} \quad j = 1, \ldots, p.
\]

But \( \varphi_a(y) \leq \varphi_a(x) \rightarrow \varphi_a(y) = \varphi_{i_1}(y_1) + \cdots + \varphi_{i_p}(y_p) \). Hence there exists \( j_0 \) such that \( \alpha \leq \beta_{j_0} \). Since \( \alpha = \beta_{j_0} \) and \( \alpha > \beta_j \) for \( j \neq j_0 \), we have \( \varphi_a(x) \rightarrow \varphi_a(y) = \varphi_a(x_{j_0}) \). From the fact that \( \varphi_a \) is a monomorphism, it is now easy to show that \( x \rightarrow y = x_{j_0} \).

The following property of \( \varphi_a \) will be needed in §3. Note that the power of a set \( H \) is denoted by \( |H| \).

**Definition 1.7.** Let \( L \) and \( M \) be distributive lattices and \( m \) a cardinal. A homomorphism \( \varphi: L \rightarrow M \) is called a \( m \)-homomorphism provided:

If \( H \subseteq L \), \( 0 < |H| \leq m \), and \( \Sigma_v(H) \) exists then \( \Sigma_v h(H) \) exists and equals \( h(\Sigma_v(H)) \); and similarly for products. The homomorphism is complete if it is a \( m \)-homomorphism for each cardinal \( m \).

**Lemma 1.8.** Each monomorphism \( \varphi_a: L_a \rightarrow L(P) \) of \( ([\varphi_a]_{\alpha \in S}, L(P)) \) is complete.

**Proof.** Let \( H \subseteq L_a \) and suppose \( x = \Sigma_{L_a}(H) \) exists. Clearly \( \varphi_a(y) \leq \varphi_a(x) \) for all \( y \in H \). Now suppose that \( \Sigma_{L(P)}(H_1) \cdots \Sigma_{L(P)}(H_n) \) is an upper bound for \( \varphi_a(H) \), where \( H_i \subseteq \cup_{\alpha \in S} \varphi_a(L_a) \) for \( i = 1, \ldots, n \).
We can assume \( H_1 = \{ \varphi_a(\chi_i \psi_{a_j}) \mid x_i \in L_{a_i} \text{ and } \alpha_k \neq \alpha_j \text{ for } k \neq j \} \). Suppose:

- (1.15) there exists \( j \in \{1, \ldots, m\} \) such that \( \alpha < \alpha_j \). Then \( \varphi_a(x) < \varphi_a(\chi_j) \) so
  \[
  \varphi_a(x) \leq \Sigma_{L(P)}(H_j). \tag{1.16}
  \]

Now suppose that (1.15) does not hold. Since \( \varphi_a(y) \leq \varphi_a(x_i) + \cdots + \varphi_{a_m}(x_m) \) for each \( y \in H \), and \( \alpha_j \neq \alpha_k \) for \( j \neq k \), there exists \( \alpha_j \) such that \( \alpha = \alpha_j \) and \( \varphi_a(y) \leq \varphi_a(x_j) \) for all \( y \in H \). Hence \( x_i \in L_a \) and \( y \leq x_j \) for all \( y \in S \). So \( x \leq x_j \) and therefore (1.16) is valid regardless of the validity of (1.15). Applying this argument to each \( H_i \), we have \( \varphi_a(x) \leq \Sigma_{L(P)}(H_1) \cdots \Sigma_{L(P)}(H_m) \), and so \( \varphi_a(\Sigma_{L_a}(H)) = \Sigma_{L(P)}\varphi_a(H) \). Similarly for products.

2. \((J, M, m)\)-extensions. Throughout this section, let \( L \) be a distributive lattice, and \( m \) a fixed infinite cardinal. Also let \( J \) and \( M \) be collections of nonempty subsets of \( L \) such that:

- (2.1) \( |H| \leq m \) for each \( H \in J \) and each \( H \in M \).
- (2.2) \( \Sigma_L(H) \) exists for each \( H \in J \) and \( \Pi_L(H) \) exists for each \( H \in M \).

**Definition 2.1.** If \( L' \) is a distributive lattice then a homomorphism \( f: L \to L' \) is a \((J, M)\)-homomorphism provided:

- (2.3) If \( H \in J \) then \( \Sigma_Lf(H) \) exists and equals \( f(\Sigma_L(H)) \).
- (2.4) If \( H \in M \) then \( \Pi_Lf(H) \) exists and equals \( f(\Pi_L(H)) \).

**Definition 2.2.** The pair \((\psi, E)\) is called a \((J, M, m)\)-extension of \( L \) provided:

- (2.5) \( E \) is a \( m \)-complete distributive lattice.
- (2.6) \( \psi: L \to E \) is a \((J, M)\)-monomorphism.
- (2.7) \( \psi(L) \) \( m \)-generates \( E \) (i.e., \( E \) is the smallest \( m \)-complete sublattice of \( E \) that contains \( \psi(L) \)).

Every distributive lattice has a \((\phi, \phi, m)\)-extension; the smallest \( m \)-ring of subsets of the Stone space \( X \) of \( L \) that contains all of the compact-open sets of \( X \), together with the correspondence that associates elements of \( L \) with compact-open sets of \( X \). If \( J(M) \) is the collection of all subsets of \( L \) of power \( \leq m \) which have a sum (product) in \( L \) then a \((J, M, m)\)-extension of \( L \) is called a \( m \)-regular extension. Note that in this case, \( \psi \) is a \( m \)-homomorphism. In [5], Crawley has constructed an example of a distributive lattice which can not be regularly imbedded in any complete distributive lattice. In this example if we take \( I \) to be countable then \( L \) will have no \( \aleph_0 \)-regular extension.

A sufficient condition for \( L \) to have a \((J, M, m)\)-extension is that \( L \) be conditionally implicative. Indeed, it is easily verified that the MacNeille completion [3, p. 58] of such a lattice is also conditionally implicative and hence distributive. Note that the category of condi-
tionally implicative lattices includes the categories of Boolean algebras, chains, free and finite distributive lattices, and pseudo Boolean algebras.

Another sufficient condition for $L$ to have a $(J, M, m)$-extension is that

\begin{equation}
y \sum_{i \in I} x_i = \sum_{i \in I} y x_i \text{ and } y + \prod_{i \in I} x_i = \prod_{i \in I} (y + x_i)
\end{equation}

whenever the left sides exist and $|I| \leq m$. This follows from [4, Lemma 2].

If $(\psi, E)$ and $(\psi', E')$ are $(J, M, m)$-extensions of $L$, then we write

\begin{equation}(\psi, E) \leq (\psi', E')\end{equation}

provided there is a $m$-homomorphism $h: E' \to E$ such that $h \psi' = \psi$. Clearly $h$ is onto. If $h$ is an isomorphism we say $(\psi, E)$ is isomorphic with $(\psi', E')$. Isomorphism in this sense is an equivalence relation $\simeq$, and by identifying isomorphs, (2.9) determines a partial ordering on the equivalence classes of $K/\simeq$ where $K$ is the set of $(J, M, m)$-extensions of $L$.

By generalizing the method in [6, p. 166], we now investigate the class $K$.

**Definition 2.3.** A congruence relation $R$ on a $m$-complete lattice $M$ is called a $m$-congruence relation on $M$ if whenever $I$ is an index set of power $\leq m$ and $(x_i, y_i) \in R$ for each $i \in I$ then

$$(\Sigma_{i \in I} x_i, \Sigma_{i \in I} y_i) \in R \text{ and } (\Pi_{i \in I} x_i, \Pi_{i \in I} y_i) \in R.$$ 

For a $m$-congruence relation $R$ on a $m$-complete lattice $M$, let $[x]_R$ be the equivalence class containing $x \in M$, and let 

$$M/R = \{[x] | x \in M\}.$$ 

The following theorem is easily verified.

**Theorem 2.4.** If $R$ is a $m$-congruence relation on a $m$-complete lattice $M$ then $M/R$ is partially ordered as follows: $[x]_R \leq [y]_R$ provided there exists $x', y' \in M$ such that $(x, x') \in R, x' \leq y'$ and $(y', y) \in R$. Furthermore, $M/R$ is a $m$-complete lattice such that if $H \subseteq M$ and $0 < |H| \leq m$ then $\Sigma_{x \in H} [x]_R = [\Sigma_{x \in H}]_R$ and $\Pi_{x \in H} [x]_R = [\Pi_{x \in H}]_R$. If $M$ is distributive so is $M/R$.

Let $n$ be the power of the distributive lattice $L$ and let $F$ be the free $m$-complete distributive lattice with $n$ generators. That is, $F$ satisfies:

\begin{equation}F\text{ is a } m\text{-complete distributive lattice and is } m\text{-generated by a subset } G\text{ of power } n.\end{equation}

\begin{equation}\text{If } h: G \to M\text{ is a function, where } M\text{ is a } m\text{-complete}
\end{equation}
distributive lattice, then \( h \) can be extended to a \( m \)-homomorphism on \( F \).

By (2.11), \( G \) has the property that if \( G_1, G_2 \) are finite nonempty subsets of \( G \) and \( \Pi_F(G_i) \subseteq \Sigma(G_2) \), then \( G_1 \cap G_2 \neq \emptyset \). So the sublattice \( F' \) generated by \( G \) is freely generated by \( G \), and there is an epimorphism \( g : F' \to L \). Let \( R \) be the set of \( m \)-congruence relations \( R \) on \( F \) such that:

(2.12) If \( x, y \in F' \) then \( (x, y) \in R \Rightarrow g(x) = g(y) \).

(2.13) If \( H \subseteq F', \ |H| \leq m, g(H) \in J, x \in F' \), and \( g(x) = \Sigma_L g(H) \) then \( (x, \Sigma_L g(H)) \in R \).

(2.14) If \( H \subseteq F', \ |H| \leq m, g(H) \in M, x \in F' \) and \( g(x) = \Pi_L g(H) \) then \( (x, \Pi_L g(H)) \in R \).

For each \( R \in R \), let \( F'_R \) be the sublattice \( \{[x]_R \mid x \in F'\} \) of \( F/R \).

By (2.12), the mapping \( g_R : F'_R \to L \) defined by:

(2.15) \( g_R([x]_R) = g(x) \)

for each \( x \in F' \) is an isomorphism. Define \( \psi_R : L \to F'/R \) by \( \psi_R = i_R g_R^{-1} \) where \( i_R : F'_R \to F/R \) is the inclusion map. We have

(2.16) \( \psi_R g(x) = [x]_R \) for each \( x \in F' \).

**Theorem 2.5.** For each \( R \in R \), the pair \((\psi_R, F'/R)\) is a \((J, M, m)\)-extension of \( L \).

**Proof.** First \( F'/R \) is \( m \)-complete by Theorem 2.4. Let \( G \in J \), then \( |G| \leq m \) and \( \Sigma_L(G) \) exists. Since \( g \) is onto \( L \) there exists \( \{x\} \cup H \subseteq F' \) such that \( |H| \leq m, g(H) = G \) and \( g(x) = \Sigma_L g(H) \). By (2.13), \( (x, \Sigma_L g(H)) \in R \) so

\[
\psi_R(\Sigma_L(G)) = [x]_R = [\Sigma_L(H)]_R = \Sigma_{F'/R}[y]_R \mid y \in H
\]

\[
= \Sigma_{F'/R} \psi_R g(H) = \Sigma_{F'/R} \psi_R(G).
\]

A similar argument for \( G \in M \) implies that \( \psi_R \) is a \((J, M)\)-monomorphism. Finally since

\[
\psi_R(L) = \psi_R g(F') = F'_R
\]

and \( F' \) \( m \)-generates \( F \), we have \( \psi_R(L) \) \( m \)-generates \( F'/R \).

**Theorem 2.6.** For each \((J, M, m)\)-extension \((\psi, E)\) of \( L \), there exists \( R \in R \) such that \((\psi, E) \cong (\psi_R, F'/R)\).

**Proof.** By (2.11), the mapping \( \psi g : F' \to E \) can be extended to a \( m \)-homomorphism \( k \) of \( F \) onto \( E \). Define a relation \( R \) on \( F \) by \( (x, y) \in R \) if \( k(x) = k(y) \). It is easily verified that \( R \in R \) so that by Theorem 2.5, \((\psi_R, F'/R)\) is a \((J, M, m)\)-extension of \( L \). Next, define \( h : F'/R \to E \) by \( h([x]_R) = k(x) \) for each \( x \in F' \). Then \( h \) is an isomorphism. Let \( y \in L \), then there is an \( x \in F' \) such that \( g(x) = y \), so
\[ h \varphi_R(y) = h \psi_R g(x) = h([x]_R) = k(x) = \psi g(x) = \psi(y). \]

It follows that \((\psi, E) \simeq (\psi_R, F/R)\).

**Theorem 2.7.** If \((\psi_R, F/R)\) and \((\psi_{R'}, F'/R')\) are \((J, M, m)\)-extensions of \(L\) then

\[(\psi_R, F/R) \leq (\psi_{R'}, F'/R')\]

if and only if

\[ R' \subseteq R. \]

Consequently, \(K/\simeq\) is isomorphic with \(R\) (partially ordered by the converse of inclusion).

**Proof.** Suppose there is a \(m\)-epimorphism \(h : F/R' \to F/R\) such that \(h \psi_{R'} = \psi_R\). For each \(x \in F'\), \(h([x]_{R'}) = h \psi_R g(x) = \psi_R g(x) = [x]_R\).

But, in fact, \(\{x \in F' \mid h([x]_{R'}) = [x]_R\}\) is a \(m\)-sublattice of \(F\) containing \(F''\). So \(h([x]_{R'}) = [x]_R\) for each \(x \in F'\). Thus if \((x, y) \in R'\) then \([x]_R = h([x]_{R'}) = h([y]_{R'}) = [y]_R\), i.e., \(R' \subseteq R\). For the converse, define \(h : F'/R' \to F/R\) by \(h([x]_{R'}) = [x]_R\) for each \(x \in F\). The hypothesis implies \(h\) is a \(m\)-homomorphism. Since \(h \psi_{R'} = \psi_R\), the result follows.

**Corollary 2.8.** The intersection \(\rho = \bigcap_{R \subseteq R} R\) is an element of \(R\) and hence the equivalence class containing \((\psi, F/\rho)\) is the greatest element in \(K/\simeq\). Here it is assumed \(R \neq \emptyset\).

**Proof.** Conditions (2.12), (2.13), and (2.14) are satisfied by \(\rho\).

**Definition 2.9.** A \((J, M, m)\)-extension \((\psi, E)\) of \(L\) is said to be **free** provided that for each \(m\)-complete distributive lattice \(L'\) and each \((J, M)\)-homomorphism \(f : L \to L'\), there exists a \(m\)-homomorphism \(h : E \to L'\) such that \(f = h \psi\).

The main result of this section is then:

**Theorem 2.10.** If \(L\) has a \((J, M, m)\)-extension then \(L\) has a free \((J, M, m)\)-extension: \((\psi, F/\rho)\).

**Proof.** As in the proof of Theorem 2.6, the mapping \(f g : F' \to L'\) can be extended to a \(m\)-homomorphism \(h' : F \to L'\). Define a relation \(R'\) on \(F\) by \((x, y) \in R'\) if \(h'(x) = h'(y)\). We first show that \(R' \cap \rho \subseteq R\). Clearly \(R' \cap \rho\) is a \(m\)-congruence relation. For (2.12), (2.13), and (2.14), first let \(x, y \in F'\). Since \(\rho \subseteq R\), \((x, y) \in R' \cap \rho\) implies \(g(x) = g(y)\).

Conversely if \(g(x) = g(y)\) then \(f g(x) = f g(y)\) so \((x, y) \in \rho \cap R'\). If
H \subseteq F', |H| \leq m, g(H) \in J, x \in F' and g(x) = \Sigma_L g(H) then since \rho \in R, (x, \Sigma_L g(H)) \in \rho. But f is a \((J, M)\)-homomorphism so fg(x) = f(\Sigma_L g(H)) = \Sigma_L fg(H). Hence h'(x) = \Sigma_L h'(H) = h'(\Sigma_L g(H)), i.e., (x, \Sigma_L g(H)) \in \rho \cap R'. Similarly for (2.14). Now \rho \cap R' \in R so \rho \subseteq R'. Hence we can define h : F/\rho \rightarrow L' by h([x]_\rho) = h'(x) for each x \in F. It follows that h is a \(m\)-homomorphism and f = h\psi_\rho.

### 3. \(m\)-order sums

In this section \(\{L_a\}_{a \in S}\) is a fixed set of distributive lattices, \(m\) is a fixed infinite cardinal and \(P\) is a partial ordering on \(S\).

**DEFINITION 3.1.** The pair \(\langle \{\psi_a\}_{a \in S}, E \rangle\) is said to be a \(m\)-order sum of \(\{L_a\}_{a \in S}\) over \(P\) provided \(E\) is a \(m\)-complete distributive lattice, and for each \(a \in S\), \(\psi_a : L_a \rightarrow E\) is a \(m\)-monomorphism such that:

1. \(E\) is \(m\)-generated by \(\bigcup_{a \in S} \psi(L_a)\).
2. If \(a < b\) then \(\psi_a(x) < \psi_b(y)\) for each \(x \in L_a\) and \(y \in L_b\).
3. If \(L'\) is a \(m\)-complete distributive lattice and \(\{f_a : L_a \rightarrow L'\}_{a \in S}\) is a collection of \(m\)-homomorphisms such that whenever \(a < b\) then \(f_a(x) \leq f_b(y)\) for all \(x \in L_a, y \in L_b\), then there exists a \(m\)-homomorphism \(f : E \rightarrow L'\) such that \(f\psi_a = f_a\) for each \(a \in S\).

It follows that the \(m\)-order sum is essentially unique—if it exists. Note also that if \(P\) is the trivial ordering on \(S\) and \(|L_a| = 1\) for each \(a \in S\) then \(E\) is the free \(m\)-complete distributive lattice with \(|S|\) generators. We now investigate the existence question.

Let \(\langle \{\phi_a\}_{a \in S}, L(P) \rangle\) be the order sum of \(\{L_a\}_{a \in S}\) over \(P\). Let \(J\) be the class of all sets of the form \(\phi_a(H)\) where

\[
\alpha \in S, H \subseteq L_a, |H| \leq m, H \neq \phi \text{ and such that } \Sigma_{L_a}(H) \text{ exists.}
\]

Let \(M\) be the class of all sets of the form \(\phi_a(H)\) satisfying (3.4) and such that \(I_{L_a}(H)\) exists. Note that since \(\phi_a\) is a complete monomorphism (Lemma 1.8), conditions (2.1) and (2.2) of §2 are satisfied.

**THEOREM 3.2.** If \(L(P)\) has a \((J, M, m)\)-extension then \(\{L_a\}_{a \in S}\) has a \(m\)-order sum over \(P\).

**Proof.** By Theorem 2.10, \(L(P)\) has a free \((J, M, m)\)-extension \((\psi, E)\). We will show that \(\langle \{\psi \phi_a\}_{a \in S}, E \rangle\) is the required \(m\)-order sum. Let \(H \subseteq L_a, 0 < |H| \leq m\) and suppose that \(\Sigma_{L_a}(H)\) exists. Then \(\phi_a(H) \in J\). Since \(\psi\) is a \((J, M)\)-monomorphism and \(\phi_a\) is complete,

\[
\psi \phi_a(\Sigma_{L_a}(H)) = \Sigma_E \psi \phi_a(H).
\]

Similary for products. So \(\psi \phi_a\) is a \(m\)-monomorphism. Since \(\bigcup_{a \in S} \phi_a(L_a)\) generates \(L(P)\) and \(\psi(L(P))\) \(m\)-generates \(E\), it follows that
Finally, let \( L' \) be a \( m \)-complete distributive lattice and \( \{ f_\alpha : L_\alpha \rightarrow L' \}_{\alpha \in S} \) a family of \( m \)-homomorphisms with the property that \( \alpha < \beta \) implies \( f_\alpha(x) \leq f_\beta(y) \) for all \( x \in L_\alpha, y \in L_\beta \). By (1.3) three exists a homomorphism \( f' : L(P) \rightarrow L' \) such that \( f'_\alpha = f_\alpha \) for each \( \alpha \in S \). Since \( \varphi_\alpha \) is complete, \( f' \) is a \((J,M)\)-homomorphism. But \((\varphi,E)\) is a free-\((J,M)\)-extension, so there exists a \( m \)-homomorphism \( f : E \rightarrow L' \) such that \( f' = f\varphi \). Thus \( f\varphi_\alpha = f_\alpha \) for each \( \alpha \in S \).

**Corollary 3.3.** If \( \{L_\alpha\}_{\alpha \in S} \) is a collection of conditionally implicative lattices (or lattices satisfying (2.8)), then \( \{L_\alpha\}_{\alpha \in S} \) has a \( m \)-order sum over \( P \) for each partial ordering \( P \) on \( S \).

**Proof.** This is immediate from Theorem 3.2 and the remarks following Definition 2.2.

A necessary condition for the \( m \)-order sum \( \{\varphi_\alpha\}_{\alpha \in S}, E \) over \( P \) to exist is that each \( L_\alpha \) have a free \( m \)-regular extension (consider the smallest \( m \)-complete sublattice of \( E \) that contains \( \varphi_\alpha(L_\alpha) \)). A case in which an \( m \)-order sum has a rather simple structure is obtained in the next theorem. For the definition of ordinal sum, see [2, Definition 1.3].

**Theorem 3.4.** Suppose \( S \) is finite and \( P \) is a chain in \( P \). If \((\varphi_\alpha, E_\alpha)\) is a free \( m \)-regular extension of \( L_\alpha \) for each \( \alpha \in S \), then \( \{i_\alpha \varphi_\alpha\}_{\alpha \in S}, E \) is the \( m \)-order sum of \( \{L_\alpha\}_{\alpha \in S} \) over \( P \), where \( E \) is the ordinal sum of \( \{E_\alpha\}_{\alpha \in S} \) and \( i_\alpha : E_\alpha \rightarrow E \) is the inclusion map for each \( \alpha \in S \).

**Proof.** We can assume that \( S = \{1, 2, \ldots, n\} \) with the usual ordering and \( \{E_\alpha\}_{\alpha \in S} \) is a pair-wise disjoint family. Clearly, for \( H \subseteq E, 0 < |H| < m \), we have \( \Sigma_\alpha(H) = \Sigma_\beta(H \cap E_\beta) \) where \( \beta = \max \{\alpha \in S \mid H \cap E_\alpha \neq \emptyset\} \). It is evident that \( E \) is a \( m \)-complete distributive lattice, \( m \)-generated by \( \bigcup_{\alpha \in S} i_\alpha \varphi_\alpha(L_\alpha) \). Now assume the hypothesis of (3.3). Since \((\varphi_\alpha, E_\alpha)\) is a \( m \)-regular extension of \( L_\alpha \), there exists a \( m \)-homomorphism \( g_\alpha : E_\alpha \rightarrow L' \) such that \( g_\alpha \varphi_\alpha = f_\alpha \) for each \( \alpha \in S \). The function \( g : E \rightarrow L' \) defined by \( g(x) = g_\alpha(x) \) for \( x \in E_\alpha \) has the property \( g \varphi_\alpha = f_\alpha \) for each \( \alpha \in S \). To show \( g \) preserves order, suppose \( \alpha < \beta, x \) is a fixed element in \( L_\alpha \) and let \( F = \{y \in E_\beta \mid g_\alpha \varphi_\alpha(x) \leq g_\beta(y)\} \). Then

(i) \( \varphi_\beta(L_\beta) \subseteq F \) and

(ii) \( F \) is a \( m \)-complete sublattice of \( E_\beta \).

It follows that \( F = E_\beta \) and

\[ g_\alpha \varphi_\alpha(x) \leq g_\beta(y) \quad \text{for} \quad x \in L_\alpha, \ y \in E_\beta. \]
Now let $y$ be a fixed element of $E_{\beta}$ and let $G = \{x \in E_\alpha \mid g_\alpha(x) \leq g_\beta(y)\}$. Then

(iii) $\psi_\alpha(L_\alpha) \subseteq G$ and

(iv) $G$ is a $m$-complete sublattice of $E_\alpha$.

It follows that $G = E_\alpha$ and that for $x \in L_\alpha$, $y \in L_{\beta}$, $g(x) \leq g(y)$. Finally, to show $g$ is a $m$-homomorphism, let $H \subseteq E$, $0 < |H| < m$, and set $\beta = \max \{\alpha \in S \mid H \cap E_\alpha \neq \emptyset\}$. Then

$$\Sigma_{E\beta} g(H) \leq g(\Sigma_{E\beta}(H \cap E_\beta)) = g(\Sigma_{E\beta}(H \cap E_\beta)) = g_\beta(\Sigma_{E\beta}(H \cap E_\beta))$$

$$= \Sigma_{L\beta} g_\beta(H \cap E_\beta) \leq \Sigma_{L\beta} g(H).$$

So

$$\Sigma_{L\beta} g(H) = g(\Sigma_{E\beta}(H)).$$

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References


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