Pacific Journal of Mathematics

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Vol. 27, No. 3 March 1968

PACIFIC JOURNAL OF MATHEMATICS
Vol. 27, No. 3, 1968

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Let A be a function algebra with its maximal ideal space M_A. Let B be a function algebra such that $A \subset B \subset C(M_A)$. What can be said about M_B? We prove that $M_A = M_B$ if every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that the topological boundary bW of each W is contained in the Choquet boundary of A or if A is a normal function algebra. The first condition is satisfied if M_A is a one dimensional topological space. Let $H(A)$ be the function algebra on M_A generated by all functions which are locally approximable in A. We prove that $M_{H(A)} = M_A$ and then we try to generalize this result. If $f \in C(M_A)$ is such that f is locally approximable in A at every point where f is different from zero then M_A is the maximal ideal space of the function algebra generated by A and f. We also look at closed subsets F of M_A such that $M_{H(F)} = F$ where $H(F)$ is the function algebra generated by restricting to F all functions that are defined and locally approximable in A in some neighborhood of F. These sets are called natural sets. We prove that there exists a smallest natural set $B(F)$ containing a closed set F in M_A and that the Silov boundary of $H(B(F))$ is contained in F. We also find conditions that guarantee that a closed set in M_A is a natural set.

If X is a set and f is a complex-valued function defined on X then $|f|_V = \sup\{|f(x)| \mid x \in V\}$ for every $V \subset X$ and f_V is the restriction of f to V. If V is a subset of a topological space X then bV is the topological boundary of V in X. If A is a function algebra we denote by M_A its maximal ideal space, and S_A its Shilov boundary. A point $x \in M_A$ is a strong boundary point in A if $\{x\} = \cap P(f)$, where $P(f)$ are peak sets of A in M_A. We shall use the wellknown fact that S_A is the closure of the strong boundary points of A in M_A. If F is a closed set in M_A then $\mathrm{Hull}_A(F) = \{x \in M_A \mid |f(x)| \leqq |f|_F$ for every $f \in A\}$. If $x \in \mathrm{Hull}_A(F)$ we say thet F is a support of x. A minimal support of x is a support F of x such that no proper closed subset of F is a support of x. Now we have the principle of minimal supports. Let F be a minimal support of x. Suppose $\{f_n\} \in A$ is such that $|f_n|_F \leqq K$ for some constant K independent of n and $\lim |f_n|_{W \cap F} = 0$, where W is an open subset of M_A such that $W \cap F$ is not empty. Then it follows that $\lim f_n(x) = 0$. If F is a closed set in M_A then A_F is the function algebra on F generated by functions $f \in C(F)$ such that $f = g$ on F for some $g \in A$. Now M_{A_F} can be identified with

$\operatorname{Hull}_A(F)$. If F is a closed set in M_A such that $F = \operatorname{Hull}_A(F)$ we say that F is an A-convex set. A is a convex function algebra if every closed set in M_A is A-convex. If B is a function algebra on M_A such that $A \subset B$ then the maximal ideal space M_B contains M_A and $S_B \subset M_A$. If $x \in M_B$ there exists a point $y(x) \in M_A$ such that $f(x) = f(y(x))$ for $f \in A$. If V is a subset of M_A we put $\{V\}_B = \{x \in M_B \mid y(x) \in V\}$. The set $\{V\}_B$ is called the fiber of V in M_B. The correspondence between points x in M_A and the fibers $\{x\}_B$ is continuous in the following way: Let W be an open neighborhood of $\{x\}_B$ in M_B for some point $x \in M_A$. Then there exists a neighborhood V of x in M_A such that $\{V\}_B \subset W$. If W is an open set in M_A then $H_0(W) = \{f \in C(W) \mid f$ is locally approximable in A at every point in W, i.e., if $x \in W$ there exists a neighborhood $V \subset W$ of x and $\{g_n\} \in A$ such that $\lim |g_n - f|_V = 0.\}$. We put $H_0(A) = H_0(M_A)$ and $H(A)$ is the function algebra generated by $H_0(A)$ on M_A. If F is a closed set in M_A then $H_0(F) = \{f \in C(F) \mid f = g$ on F for some $g \in H_0(V)$, where V is some neighborhood of $F\}$. We let $H(F)$ be the function algebra on F generated by $H_0(F)$. We shall now discuss the results of this paper. The general problem which interests us here is the following: Let A be a function algebra with its maximal ideal space M_A. Let B be a function algebra such that $A \subset B \subset C(M_A)$. What can be said about M_B? In Lemma 1 we give the well-known construction which shows that M_B in general is strictly larger than M_A. A point $x \in M_A$ is a stationary point if $\{x\}_B = \{x\}$ for every B such that $A \subset B \subset C(M_A)$. A is a resistent function algebra if M_A consists of stationary points. In Theorem 2 we prove that A is a resistent function algebra if every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that $\{bW\}$ consist of stationary points. We remark here that the Choquet boundary of A is contained in the set of stationary points and that A is resistent if $M_A = [0, 1]$. A function algebra A on a compact set X is regular if A separates points from closed subsets of X. It is wellknown that if $X = M_A$ then A is normal, i.e., A separates disjoint closed sets. In Theorem 4 we prove that if A is a regular function algebra on X then X consists of stationary points when we consider X as a closed subset of M_A. We remark that if A is a normal function algebra on X then $X = M_A$. The rest of this paper is mostly devoted to a study of relations between A and $H(A)$. We have never introduced the general concept of A-holomorphic functions as is done in [3]. We wish to point out that our methods come almost entirely from [3] and [4]. Our proof of Theorem 5 uses an argument which is essentially the same as in Lemma 3.1, p. 368, in [3]. We point out that Theorem 7 gives a proof of Rado's Theorem: Let $f \in C(F)$ where F is a polynomially convex compact set in the complex plane. Assume that f is analytic if f is different from zero. Then it follows that f is analytic in the interior of F and hence $f \in P(F)$, i.e., f can be uniformly approximated by polynomials on F. In Theorem 8 we prove that if $H(A)$ is a resistent function algebra then A is a resistent function algebra. We also discuss the general problem of determining 'domains of holomorphy' in general function algebras. A closed set F in M_A is a natural set if $M_{H(F)} = F$. The main result about natural sets is contained in Theorem 10 which was essentially wellknown in [3]. Every closed subset F of M_A is contained in a smallest natural set $B(F)$, the barrier of F. We have also introduced the set $\hat{F} = \{y \in M_A \mid \{y\}_{H(F)} \cap M_{H(F)}$ is not empty$\}$. We know that $\hat{F} \subset B(F)$ and in general the inclusion is strict.[1] Theorem 12 is essentially wellknown in [5] but we believe our proof is different.

1. DEFINITION 1. A function algebra A is resistent if $M_B = M_A$ for every function algebra B such that $A \subset B \subset C(M_A)$.

LEMMA 1. *A resistent function algebra is convex.*

Proof. Let A be a function algebra such that $\mathrm{Hull}_A(F) - F$ is not empty for some closed set F in M_A. Let $B = \{g \in C(M_A) \mid g_F \in A_F\}$. Obviously $A \subset B \subset C(M_A)$ and now we prove that $M_A \neq M_B$. Let $x \in \mathrm{Hull}_A(F) - F$. If $g \in B$ we can find $\{f_n\} \in A$ such that $\lim |g - f_n|_F = 0$. Now we put $\hat{x}(g) = \lim f_n(x)$. It is easily seen that $\hat{x}$ is a well defined complex-valued homomorphism on B. Hence there exists a point $y \in M_B$ such that $\hat{x}(g) = g(y)$ for $g \in B$. In particular $f(x) = \hat{x}(f) = f(y)$ for $f \in A$. If $M_A = M_B$ it follows that $\hat{x}(g) = g(x)$ for $g \in B$. But now we choose $g \in B$ such that $g(x) = 1$ while $g = 0$ on F and obtain a contradiction. Hence $M_A \neq M_B$ and the lemma follows.

LEMMA 2. *Let A be a convex function algebra and let*

$$A \subset B \subset C(M_A) .$$

Then the fibers $\{x\}_B$ are connected in M_B for every point $x \in M_A$.

Proof. Suppose that some fiber $(x)_B$ is disconnected in M_B. Hence there exists a closed component G of $\{x\}_B$ such that $G \subset M_B - M_A$. Now we can find a closed neighborhood W of G in M_B such that $bW \cap \{x\}_B$ is empty and $W \subset M_B - M_A$. Let $F = \{y \in M_A \mid \{y\}_B \cap bW$ is not empty$\}$. Obviously F is a closed subset of M_A such that $x \notin F$. Let $y \in G$, then the local maximum principle shows that $|g(y)| \leqq |g|_{bW}$ for $g \in B$. It follows that $|f(x)| \leqq |f|_F$ for $f \in A$, hence $x \in \mathrm{Hull}_A(F)$,

[1] I am indebted to the referee for giving an example where $F \neq B(\hat{F})$.

a contradiction to the fact that A is a convex function algebra.

THEOREM 1. *Let V be a closed A-convex subset of M_A such that A_V is resistent. Let $f \in C(M_A)$ be such that $f = 0$ in $M_A - V$, then $M_{A(f)} = M_A$.*

Proof. Assume that $D = M_{A(f)} - M_A$ is not empty. Let $x \in D$ and choose a minimal support F of x such that $F \subset M_A$. Now $F \subset V$ is impossible since A_V is a resistent function algebra. Because $f = 0$ in $M_A - V$ the principle of minimal supports shows that $f(x) = 0$. Choose $y \in M_A$ such that $g(x) = g(y)$ for $g \in A$. Since y and x are different points of $M_{A(f)}$ it follows that $f(y)$ must be different from zero, hence $y \in V$. We have now proved that $D \subset \{V\}_{A(f)}$. Now Lemma 1 shows that A_V is a convex function algebra and Lemma 2 can be applied to show that $\{z\}_{A(f)}$ are connected in $M_{A(f)}$ for every $z \in V$. In particular $\{y\}_{A(f)}$ has no isolated points in $M_{A(f)}$. Since D is an open subset of $M_{A(f)}$ we can find $x_1 \in D \cap \{y\}_{A(f)}$ such that $x_1 \neq x$. But now we get $f(x_1) = f(x) = 0$ and then x and x_1 are not different points in $M_{A(f)}$, a contradiction.

DEFINITION 2. A point $x \in M_A$ is stationary if $\{x\}_B = \{x\}$ for every function algebra B such that $A \subset B \subset C(M_A)$.

THEOREM 2. *Let A be a function algebra such that every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that each bW consists of stationary points, then A is a resistent function algebra.*

Proof. Suppose that B is a function algebra such that

$$A \subset B \subset C(M_A)$$

and assume that $D = M_B - M_A$ is not empty. Let $z \in D$ and choose $y \in M_A$ such that $f(z) = f(y)$ for $f \in A$. Choose an open neighborhood V of y in M_A such that bV consists of stationary points. Let W be a closed B-convex neighborhood of z in M_B such that $W \subset D$. Now $\{V\}_B \cap W$ is open and closed in W. We apply Shilov's Idempotent Theorem to the function algebra B_W. Hence we find $\{f_n\} \in B$ such that $\lim |f_n - 1|_{W \cap \{V\}_B} = 0$ while $\lim |f_n|_{W - \{V\}_B} = 0$. Choose a minimal support F of z such that $F \subset bW$. It follows from the principle of minimal supports that $F \subset bW \cap \{V\}_B$. Now we let V shrink to y in M_A and it follows that $z \in \mathrm{Hull}_B(bW \cap \{y\}_B)$. This holds for every $z \in D \cap \{y\}_B$ when W is a closed B-convex neighborhood of z such that $W \subset D$. Now we choose a strong boundary point $x \in D \cap \{y\}_B$ of the function algebra $B_{\{y\}_B}$ to obtain a contradiction.

DEFINITION 4. A point $x \in M_A$ is locally regular if there exists a neighborhood V of x such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of y and $f(x) = 1$.

THEOREM 3. *A locally regular point is a stationary point.*

Proof. Let $x \in M_A$ be a locally regular point. Let B be a function algebra such that $A \subset B \subset C(M_A)$. Let $D = M_B - M_A$ and assume that $\{x\}_B \cap D$ is not empty. Let V be an open neighborhood of x in M_A such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of y and $f(x) = 1$. Let $z \in \{x\}_B \cap D$ and choose a closed neighborhood W of z in M_B such that $W \subset D \cap \{V\}_B$. Let F be a minimal support of z such that $F \subset bW$. It follows now that $F \subset \{x\}_B$ holds. Hence $z \in \text{Hull}_B(bW \cap \{x\}_B)$ and we obtain a contradiction if we choose a suitable point $z \in D \cap \{x\}_B$. Hence $\{x\}_B \cap D$ must be empty and it follows that x is a stationary point.

THEOREM 4. *Let A be a regular function algebra on a compact set X. Then every point $x \in X \cap M_A$ is a stationary point.*

Proof. Let $x \in X \cap M_A$ and put $R(x) = \{y \in M_A \mid \text{there exists } g \in A$ with $g = 0$ in a neighborhood of y and $g(x) = 1\}$. We shall now prove that $R(x) = M_A - \{x\}$ and then it follows from Theorem 3 that x is a stationary point. Let $y \in M_A - \{x\}$ and choose $g \in A$ such that $g(y) = 1$ and $g(x) = 0$. Let $V = \{z \in M_A \| g(z) | > 1/2\}$ and let $W = \{z \in X \| g(z) \leqq 1/2\}$. We choose $f \in A$ such that $f = 0$ on $X - W$ and $f(x) = 1$. If $z \in V$ we can choose a minimal support F of z such that $F \subset X$. Obviously $F \cap (X - W)$ is not empty and the principle of minimal supports implies that $f(z) = 0$. Hence $f = 0$ on V and $f(x) = 1$, i.e., $y \in R(x)$.

THEOREM 5. *Let F be a closed subset of M_A and let $f \in CM_A$ be such that f is locally approximable in A at every point in $M_A - F$. Then $M_{A(f)} - M_A \subset \{\text{Hull}_A(F)\}_{A(f)}$.*

Proof. Let $D = M_{A(f)} - M_A$. Let $K = \text{Hull}_{A(f)}(bD)$ and let $C = A(f)_K$. We have $D \subset K = M_C$ and bD contains the Shilov boundary of C. Let $x \in bD$ be a strong boundary point of C. Assume that $x \in M_A - F$. Choose a closed neighborhood V of x in M_A such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K \| h(x) | \geqq 1/2\} \subset \{V\}_{A(f)}$. Let

$$D_1 = \{x \in D \| h(x) | > 1/2\} .$$

The topological boundary bD_1 of D_1 in K is obviously contained in the set $T = \{x \in bD \| h(x) | \geqq 1/2\} \cup \{x \in K \| h(x) | = 1/2\}$. Choose a point

$x_1 \in D_1$. Now the local maximum principle shows that we can find a minimal support F of x_1 in C such that $F \subset T$. Since $|h(x_1)| > 1/2$ it follows that $F \cap bD$ contains an open subset of F. Since $F \subset T \subset \{V\}_{A(f)}$ we have $|g|_F \leqq |g|_V$ for $g \in A$. Now $\lim |g_n - f|_{F \cap bD} \leqq \lim |g_n - f|_V = 0$ and the principle of minimal supports shows that $\lim g_n(x_1) = f(x_1)$ holds. Now we also have $x_1 \in \{y_1\}_{A(f)}$ for some point $y_1 \in V$. Hence $f(y_1) = \lim g_n(y_1) = g_n(x_1) = f(x_1)$ and then x_1 and y_1 cannot be different points in $M_{A(f)}$, a contradiction. We have now proved that every strong boundary point of C must belong to F. It follows that $S_C \subset F$ and hence $M_{A(f)} - M_A \subset \mathrm{Hull}_{A(f)}(F)$. This implies that $M_{A(f)} - M_A \subset \{\mathrm{Hull}_A(F)_{A(f)}$.

LEMMA 3. *Let A be a function algebra on a compact set X. Let F be a closed subset of X. Then there exists a point $x \in F$ such that if m is a representing measure of x in A with $m(F) = 1$ then $m = e_x$, i.e., m is the unit point mass at x.*

Proof. Choose a strong boundary point $x \in F$ of the function algebra A_F.

THEOREM 6. *Let $A \subset B \subset C(M_A)$. Let $f \in B$ be such that $f \in H_0(A)$. Then f is constant on each fiber $\{x\}_B$ for $x \in M_A$.*

Proof. If $x \in M_B$ we denote by $y(x)$ the point in M_A such that $x \in \{y(x)\}_B$. Let $d(x) = |f(x) - f(y(x))|$ and assume that $d(x)$ is different from zero. Let $F = \{x \in M_B \,|\, d(z) = ||d|| = \sup d(z)\}$. Obviously F is a closed subset of M_B and $F \cap M_A$ is empty. Let $x \in F$ and choose an open neighborhood V of $y(x)$ in M_A such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Choose now a closed neighborhood W of x in M_B such that $W \subset \{V\}_B \cap (M_B - M_A)$. Let T be a minimal support of x such that $T \subset bW$. Now we can find a positive measure on T such that $g(x) = \int g dm$ from $g \in B$. It follows that $|f(x) - g_n(y(x))| = |f(x) - g_n(x)| \leqq \int |f - g_n| dm$ for every n. Hence we also get

$$|f(x) - f(y(x))| \leqq \int |f(z) - f(y(z))| dm(z) .$$

It follows that $|f(z) - f(y(z))| = ||d||$ for every $z \in T$, hence $T \subset F$. We have now proved that $x \in \mathrm{Hull}_B(bW \cap F)$ for every $x \in F$ and every closed neighborhood W of x such that $W \subset (M_B - M_A)$. Now we derive a contradiction from Lemma 3.

THEOREM 7. *Let $f \in C(M_A)$ and suppose that f is locally approximable in A at every point where f is different zero. Then $M_{A(f)} = M_A$ and $\mathrm{Hull}_A(F) = \mathrm{Hull}_{A(f)}(F)$ for every closed subset F of M_A.*

Proof. Let F be a closed subset of M_A such that $F = \text{Hull}_{A(f)}(F)$. Let us put $G = \text{Hull}_A(F)$ and assume that $D = G - F$ is not empty. Let $C = A(f)_G$. We see that the Shilov boundary S_C of C meets D. Hence we can find $x \in D$ such that x is a strong boundary point of C. Let us assume that $f(x) \neq 0$. Choose a closed neighborhood $V \subset (M_A - F)$ of x in M_A such that there exist $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Now we choose $h \in C$ such that if $P(h) = \{x \in G \mid h(x) = |h|_G\}$ then $x \in P(h)$ and $P(h) \subset V$ with $P(h) \cap bV$ empty. Since $h \in C$ we can find $\{h_n\} \in A$ with $\lim |h_n - h|_{V \cap G} = 0$. Now the local maximum principle shows that $|g(x)| \leqq |g|_{bV \cap G}$ for $g \in A$. It follows that $|h(x) = \lim |h_n(x)| \leqq \lim |h_n|_{bV \cap G} = |h|_{bV \cap G}$, contradiction to the fact that $P(h) \cap bV$ is empty. Hence we have proved that if $x \in D$ is a strong boundary point of C then $f(x) = 0$. If $x \in D$ we can choose a minimal support T of x such that $T \subset S_C$. Since $F = \text{Hull}_{A(f)}(F)$ it follows that $T \cap D$ is not empty. Since $f = 0$ on $S_C \cap D$ it follows from the principle of minimal supports that $f(x) = 0$. Hence we have proved that $f = 0$ on D. But then $A(f)_D = A_D$ and it follows easily that D cannot contain any strong boundary point of C. Hence $S_C \subset F$ which shows that D must be empty. We have now proved that $\text{Hull}_A(F) = \text{Hull}_{A(f)}(F)$ for every closed subset F of M_A. In particular we see that $Z(f) = \{x \in M_A \mid f(x) = 0\}$ is an A-convex set and using Theorem 5 it follows easily that $M_A = M_{A(f)}$.

COROLLARY 1. *$M_A = M_{H(A)}$ and $\text{Hull}_A(F) = \text{Hull}_{H(A)}(F)$ for every closed subset F of M_A.*

THEOREM 8. *If $H(A)$ is a resistent function algebra then A is a resistent function algebra.*

Proof. If A is not a resistent function algebra we can find $g_1 \cdots g_k \in C(M_A)$ such that $g_1 \cdots g_k$ have no common zero on M_A while $g_i(z) = \cdots = g_k(z) = 0$ for some point $z \in M_{A(g_1 \cdots g_k)}$. Because $H(A)$ is resistent we can find $h_1 \cdots h_k$, where each h_i is a polynomial in $g_1 \cdots g_k$ with coefficients in $H_0(A)$, such that $|h_1 g_1 + \cdots + h_k g_k - 1|_{M_A} < 1/2$. Let $h_i = \sum f_{iv} g^v$, where v runs over a finite set of multi-indices $(v_1 \cdots v_k)$ and $g^v = g_1^{v_1} \cdots g_k^{v_k}$. Each $f_{iv} \in H_0(A)$ and we define f_{iv} on $M_{A(g_1 \cdots g_k)}$ by letting f_{iv} be constant on each fiber of $M_{A(g_1 \cdots g_k)}$ over points of M_A. Each g^v is defined on $M_{A(g_1 \cdots g_k)}$ in the usual way. In this way we can extend each h_i to $M_{A(g_1 \cdots g_k)}$. Call these extensions $H_1 \cdots H_k$. It is easily seen that $H = H_1 g_1 + \cdots + H_k g_k$ is locally approximable in $A(g_1 \cdots g_k)$ on $M_{A(g_1 \cdots g_k)}$. Now $H(z) = 0$ while $|H - 1|_{M_A} < 1/2$ and since M_A contains the Shilov boundary of $A(g_1 \cdots g_k)$ we derive a contradiction from Corollary 1.

THEOREM 9. *Let $f \in C(M_A)$ be such that $f^n + a_1 f^{n-1} + \cdots + a_n = 0$*

on M_A *where* $a_1 \cdots a_n \in A$, *then* $M_A = M_{A(f)}$.

Proof. Let $g = nf^{n-1} + (n-1)a_1 f^{n-2} + \cdots + a_{n-1}$. It is well known that f is locally approximable in A at every point $x \in M_A$ where $g(x)$ is different from zero. (See [1], Th. 3.2.5, p. 71.) It follows that g is locally approximable in A at every point where g is different from zero. Now Theorem 7 shows that $Z(g)$ is A-convex and then Theorem 5 shows that $M_{A(f)} - M_A \subset \{Z(g)\}_{A(f)}$. Let us put $B = A_{Z(g)}$, then $M_B = Z(g)$ and the restriction of f to M_B satisfies the equation $nf^{n-1} + (n-1)b_1 f^{n-2} + \cdots + b_{n-1} = 0$ where $b_i \in B$ are the restrictions of a_i to $Z(g)$. Since $M_{A(f)} - M_A \subset \{Z(g)\}_{A(f)}$ we see that $M_{B(f)} - M_B$ is not empty if $M_{A(f)} - M_A$ is not empty. Hence we can use induction over n to prove that $M_{A(f)} = M_A$.

Let A be a function algebra. If F is a closed subset of M_A we have defined the function algebra $H(F)$. We are now interested in the maximal ideal space of $H(F)$.

DEFINITION. If F is a closed subset of M_A we put $\hat{F} = \{y \in M_A \mid \{y\}_{H(F)} \cap M_{H(F)}$ is not empty$\}$.

DEFINITION. A natural set in M_A is a closed subset F of M_A such that $F = M_{H(F)}$.

LEMMA 4. $(\cap F_a)^\wedge \subset \cap \hat{F}_a$ *for every family* $\{F_a\}$ *of closed subsets of* M_A.

Proof. Let $y \in M_A$ be such that $y \in (\cap F_a)^\wedge$. Hence there exists a complex-valued homomorphism C of $H(\cap F_a)$ such that $C(g) = g(y)$ for $g \in A$. If $f \in H(F_a)$ the restriction of f to $\cap F_a$ obviously gives an element of $H(\cap F_a)$. Hence C can be restricted to $H(F_a)$ and we obtain a complex-valued homomorphism of $H(F_a)$ such that $C(g) = g(y)$ for $g \in A$.

THEOREM 10. *Let* F *be a closed subset of* M_A *such that* $F = \hat{F}$, *then* $M_{H(F)} = F$.

Proof. Let $f \in H_0(F)$ and define $d(x) = |f(x) - f(y(x))|$ on $M_{H(F)}$ where $y(x)$ is the point in F such that $g(x) = g(y(x))$ for $g \in A$. Assume that d is not identical zero. Let $D = \{x \in M_{H(F)} \mid d(x) > 0\}$. Obviously $D \cap F$ is empty and hence D lies off the Shilov boundary of $H(F)$. Hence $D \subset K = \text{Hull}_{H(F)}(bD)$. Let us put $C = H(F)_K$ and choose $x \in bD$ such that x is a strong boundary point of C. Choose a closed neighborhood V of $y(x)$ in M_A such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_{V \cap F} = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K \mid |h(x)| \geqq 1/2\} \subset \{V \cap F\}_{H(F)}$. Now we obtain a contradiction using the same argument as in the final part of Theorem 5. Hence we have proved that if $f \in H_0(F)$ then f is constant on each fiber $\{x\}_{H(F)}$ when $x \in F$. Since $H_0(F)$ is a dense subalgebra of $H(F)$ it follows that $F = M_{H(F)}$.

COROLLARY 2. *If $\{F_a\}$ is a family of natural set of M_A then $\cap F_a$ is a natural set.*

Proof. Lemma 4 shows that $(\cap F_a)^\wedge \subset \cap \hat{F}_a = \cap F_a$ and then Theorem 10 implies that $\cap F_a$ is a natural set.

DEFINITION. If F is a closed subset of M_A then $B(F)$ is the intersection of all natural sets containing F. $B(F)$ is called the barrier of F.

Corollary 2 shows that $B(F)$ is the smallest natural set containing a closed subset F of M_A.

LEMMA 5. *Let F be a natural set. Let $f \in H(F)$ and let $F_1 = \{x \in F \mid |f(x)| \leqq 1\}$. Then F_1 is a natural set.*

Proof. Let $z \in M_{H(F_1)}$. If $g \in H(F)$ the restriction of g to F_1 gives an element of $H(F_1)$. It follows that $g(z) = g(y)$ for some point $y \in M_{H(F)}$ when $g \in H(F)$. In particular $f(z) = f(y)$ and since $|f(z)| \leqq |f|F_1$ it follows that $y \in F_1$. Hence we have proved that $F_1 = \hat{F}_1$ and now Theorem 10 implies that F_1 is a natural set.

THEOREM 11. *Let F be a closed subset of M_A. Let $S(F)$ be the Shilov boundary of $H(B(F))$. Then $S(F) \subset F$.*

Proof. Assume that $S(F)$ meets $B(F) - F$. Hence we can find $x \in B(F) - F$ such that x is a strong boundary point of $H(B(F))$. Now we can choose $f \in H(B(F))$ such that $F_1 = \{x \in B(F) \mid |f(x)| \leqq 1\}$ contains F and omits the point x.

Lemma 5 shows that F_1 is a natural set, a contradiction to the fact that $B(F)$ is the smallest natural set containing F.

We finally give some examples of natural subsets of M_A.

DEFINITION. An A-analytic polyhedron P is a closed set in M_A of the form $P = \{x \in V \mid |f_a(x)| \leqq 1$ where V is an open neighborhood of P and $\{f_a\}$ is a family in $H_0(V)\}$.

THEOREM 12. *An A-analytic polyhedron is a natural set.*

Proof. Let U be an open neighborhood of P and W a closed set containing U such that $W \subset V$. Now we can find finitely many $\{f_a\}$, say $f_1 \cdots f_k$ such that $P_1 = \{x \in W \mid |f_i(x)| \leqq 1, i = 1 \cdots k\}$ is contained in U. Now we can prove that P_1 is a natural set using the same argument as in the final part of Theorem 5. Finally we let U shrink to P and obtain natural sets $\{P_U\}$ such that $P = \cap P_U$. Now Corollary 2 shows that P is a natural set.

DEFINITION. If F is a closed subset of M_A we put $R_0(F) = \{h \in C(F) \mid h = f/g$ where $f, g \in A$ and g has no zero on $F\}$.

We let $R(F)$ be the function algebra on F generated by $R_0(F)$.

DEFINITION. If F is a closed subset of M_A we put $\mathrm{Hull}_R(F) = \{x \in M_A \mid g(x) \in g(F)$ for $g \in A\}$.

THEOREM 13. *$M_{R(F)} = \mathrm{Hull}_R(F)$ for every closed set F in M_A and if $M_{R(F)} = F$ then F is a natural set.*

Proof. If $y \in M_{R(F)}$ we choose $x \in M_A$ such that $g(y) = g(x)$ for $g \in A$. It is easily seen that $x \in \mathrm{Hull}_R(F)$ and that $(f/g)(y) = f(x)/g(x)$ when $f/g \in R_0(F)$. Since $R_0(F)$ is dense in $R(F)$ it follows that y is uniquely determined by x. Conversely if we choose $x \in \mathrm{Hull}_R(F)$ then the mapping $X; f/g \rightarrow f(x)/g(x)$ is well defined on $R_0(F)$. We have $|f(x)/g(x)| \leqq |f/g|_F$ for if $f(x) = g(x)$ while $|f/g|_F < 1$ we see that $(g - f)$ is different from zero on F and hence $(g - f)(x) \in (g - f)(F)$ is different from zero, a contradiction. Hence we can extend X to $R(F)$ and we obtain a complex-valued homomorphism on $R(F)$ such that g is mapped into $g(x)$ when $g \in A$. This proves that $M_{R(F)} = \mathrm{Hull}_R(F)$. If $M_{R(F)} = F$ then Corollary 1 can be applied to prove that F is a natural set.

Acknowledgement. I wish to express here my deep gratitude to Professor C. E. Rickart whose original research on this subject has been the source for this paper.

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Received June 30, 1967, and in revised form December, 18, 1967.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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