EXTENSIONS OF THE MAXIMAL IDEAL SPACE OF A FUNCTION ALGEBRA

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Let $A$ be a function algebra with its maximal ideal space $M_A$. Let $B$ be a function algebra such that $A \subset B \subset C(M_A)$. What can be said about $M_B$? We prove that $M_A = M_B$ if every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that the topological boundary $bW$ of each $W$ is contained in the Choquet boundary of $A$ or if $A$ is a normal function algebra. The first condition is satisfied if $M_A$ is a one dimensional topological space. Let $H(A)$ be the function algebra on $M_A$ generated by all functions which are locally approximable in $A$. We prove that $M_{H(A)} = M_A$ and then we try to generalize this result. If $f \in C(M_A)$ is such that $f$ is locally approximable in $A$ at every point where $f$ is different from zero then $M_A$ is the maximal ideal space of the function algebra generated by $A$ and $f$. We also look at closed subsets $F$ of $M_A$ such that $M_{H(F)} = F$ where $H(F)$ is the function algebra generated by restricting to $F$ all functions that are defined and locally approximable in $A$ in some neighborhood of $F$. These sets are called natural sets. We prove that there exists a smallest natural set $B(F)$ containing a closed set $F$ in $M_A$ and that the Silov boundary of $H(B(F))$ is contained in $F$. We also find conditions that guarantee that a closed set in $M_A$ is a natural set.

If $X$ is a set and $f$ is a complex-valued function defined on $X$ then $|f|_V = \sup \{|f(x)| : x \in V\}$ for every $V \subset X$ and $f_V$ is the restriction of $f$ to $V$. If $V$ is a subset of a topological space $X$ then $bV$ is the topological boundary of $V$ in $X$. If $A$ is a function algebra we denote by $M_A$ its maximal ideal space, and $S_A$ its Silov boundary. A point $x \in M_A$ is a strong boundary point in $A$ if $\{x\} = \cap P(f)$, where $P(f)$ are peak sets of $A$ in $M_A$. We shall use the well-known fact that $S_A$ is the closure of the strong boundary points of $A$ in $M_A$. If $F$ is a closed set in $M_A$ then $\text{Hull}_A(F) = \{x \in M_A : |f(x)| \leq |f|_V$ for every $f \in A\}$. If $x \in \text{Hull}_A(F)$ we say that $F$ is a support of $x$. A minimal support of $x$ is a support $F$ of $x$ such that no proper closed subset of $F$ is a support of $x$. Now we have the principle of minimal supports. Let $F$ be a minimal support of $x$. Suppose $\{f_n\} \in A$ is such that $|f_n|_F \leq K$ for some constant $K$ independent of $n$ and $\lim |f_n|_{W \cap F} = 0$, where $W$ is an open subset of $M_A$ such that $W \cap F$ is not empty. Then it follows that $\lim f_n(x) = 0$. If $F$ is a closed set in $M_A$ then $A_F$ is the function algebra on $F$ generated by functions $f \in C(F)$ such that $f = g$ on $F$ for some $g \in A$. Now $M_{A_F}$ can be identified with
Hull_A(F). If F is a closed set in M_A such that F = Hull_A(F) we say that F is an A-convex set. A is a convex function algebra if every closed set in M_A is A-convex. If B is a function algebra on M_A such that A ⊆ B then the maximal ideal space M_B contains M_A and S_B ⊆ M_A. If x ∈ M_B there exists a point y(x) ∈ M_A such that f(x) = f(y(x)) for f ∈ A. If V is a subset of M_A we put \{V\}_B = \{x ∈ M_B | y(x) ∈ V\}. The set \{V\}_B is called the fiber of V in M_B. The correspondence between points x in M_A and the fibers \{x\}_B is continuous in the following way: Let W be an open neighborhood of \{x\}_B in M_B for some point x ∈ M_A. Then there exists a neighborhood V of x in M_A such that \{V\}_B ⊆ W. If W is an open set in M_A then H_0(W) = \{f ∈ C(W) | f is locally approximable in A at every point in W, i.e., if x ∈ W there exists a neighborhood V ⊆ W of x and \{g_n\} ∈ A such that \lim |g_n - f|_V = 0\}. We put H_0(A) = H_0(M_A) and H(A) is the function algebra generated by H_0(A) on M_A. If F is a closed set in M_A then H_0(F) = \{f ∈ C(F) | f = g on F for some g ∈ H_0(V), where V is some neighborhood of F\}. We let H(F) be the function algebra on F generated by H_0(F). We shall now discuss the results of this paper. The general problem which interests us here is the following: Let A be a function algebra with its maximal ideal space M_A. Let B be a function algebra such that A ⊆ B ⊆ C(M_A). What can be said about M_B? In Lemma 1 we give the well-known construction which shows that M_B in general is strictly larger than M_A. A point x ∈ M_A is a stationary point if \{x\}_B = \{x\} for every B such that A ⊆ B ⊆ C(M_A). A is a resistant function algebra if M_A consists of stationary points. In Theorem 2 we prove that A is a resistant function algebra if every point x ∈ M_A has a fundamental neighborhood system \{W\} such that \{bW\} consist of stationary points. We remark here that the Choquet boundary of A is contained in the set of stationary points and that A is resistant if M_A = [0, 1]. A function algebra A on a compact set X is regular if A separates points from closed subsets of X. It is wellknown that if X = M_A then A is normal, i.e., A separates disjoint closed sets. In Theorem 4 we prove that if A is a regular function algebra on X then X consists of stationary points when we consider X as a closed subset of M_A. We remark that if A is a normal function algebra on X then X = M_A. The rest of this paper is mostly devoted to a study of relations between A and H(A). We have never introduced the general concept of A-holomorphic functions as is done in [3]. We wish to point out that our methods come almost entirely from [3] and [4]. Our proof of Theorem 5 uses an argument which is essentially the same as in Lemma 3.1, p. 368, in [3]. We point out that Theorem 7 gives a proof of Rado’s Theorem: Let f ∈ C(F) where F is a polynomially convex compact set in the complex plane. Assume that f is
analytic if \( f \) is different from zero. Then it follows that \( f \) is analytic in the interior of \( F \) and hence \( f \in P(F) \), i.e., \( f \) can be uniformly approximated by polynomials on \( F \). In Theorem 8 we prove that if \( H(A) \) is a resistent function algebra then \( A \) is a resistent function algebra. We also discuss the general problem of determining 'domains of holomorphy' in general function algebras. A closed set \( F \) in \( M_A \) is a natural set if \( M_{H(F)} = F \). The main result about natural sets is contained in Theorem 10 which was essentially wellknown in [3]. Every closed subset \( F \) of \( M_A \) is contained in a smallest natural set \( B(F) \), the barrier of \( F \). We have also introduced the set \( F' = \{ y \in M_A \mid \{ y \}_{H(F)} \cap M_{H(F)} \) is not empty} \}. We know that \( \hat{F} \subset B(F) \) and in general the inclusion is strict.\(^1\) Theorem 12 is essentially wellknown in [5] but we believe our proof is different.

1. **Definition 1.** A function algebra \( A \) is resistent if \( M_B = M_A \) for every function algebra \( B \) such that \( A \subset B \subset C(M_A) \).

**Lemma 1.** A resistent function algebra is convex.

*Proof.* Let \( A \) be a function algebra such that \( \text{Hull}_F(F) - F \) is not empty for some closed set \( F \) in \( M_A \). Let \( B = \{ g \in C(M_A) \mid g \in A \} \). Obviously \( A \subset B \subset C(M_A) \) and now we prove that \( M_A \neq M_B \). Let \( x \in \text{Hull}_F(F) - F \). If \( g \in B \) we can find \( \{ f_n \} \in A \) such that \( \lim | g - f_n |_F = 0 \). Now we put \( \hat{x}(g) = \lim f_n(x) \). It is easily seen that \( \hat{x} \) is a well defined complex-valued homomorphism on \( B \). Hence there exists a point \( y \in M_B \) such that \( \hat{x}(g) = g(y) \) for \( g \in B \). In particular \( f(x) = \hat{x}(f) = f(y) \) for \( f \in A \). If \( M_A = M_B \) it follows that \( \hat{x}(g) = g(x) \) for \( g \in B \). But now we choose \( g \in B \) such that \( g(x) = 1 \) while \( g = 0 \) on \( F \) and obtain a contradiction. Hence \( M_A \neq M_B \) and the lemma follows.

**Lemma 2.** Let \( A \) be a convex function algebra and let \( A \subset B \subset C(M_A) \).

Then the fibers \( \{ x \}_B \) are connected in \( M_B \) for every point \( x \in M_A \).

*Proof.* Suppose that some fiber \( \{ x \}_B \) is disconnected in \( M_B \). Hence there exists a closed component \( G \) of \( \{ x \}_B \) such that \( G \subset M_B - M_A \). Now we can find a closed neighborhood \( W \) of \( G \) in \( M_B \) such that \( bW \cap \{ x \}_B \) is empty and \( W \subset M_B - M_A \). Let \( F = \{ y \in M_A \mid \{ y \}_B \cap bW \) is not empty} \}. Obviously \( F \) is a closed subset of \( M_A \) such that \( x \notin F \). Let \( y \in G \), then the local maximum principle shows that \( | g(y) | \leq | g |_W \) for \( g \in B \). It follows that \( | f(x) | \leq | f |_F \) for \( f \in A \), hence \( x \in \text{Hull}_F(F) \),

\(^1\) I am indebted to the referee for giving an example where \( F \neq B(F) \).
a contradiction to the fact that $A$ is a convex function algebra.

**Theorem 1.** Let $V$ be a closed $A$-convex subset of $M_A$ such that $A_v$ is resistent. Let $f \in C(M_A)$ be such that $f = 0$ in $M_A - V$, then $M_{A(f)} = M_A$.

**Proof.** Assume that $D = M_{A(f)} - M_A$ is not empty. Let $x \in D$ and choose a minimal support $F$ of $x$ such that $F \subseteq M_A$. Now $F \subseteq V$ is impossible since $A_v$ is a resistent function algebra. Because $f = 0$ in $M_A - V$ the principle of minimal supports shows that $f(x) = 0$. Choose $y \in M_A$ such that $g(x) = g(y)$ for $g \in A$. Since $y$ and $x$ are different points of $M_{A(f)}$ it follows that $f(y)$ must be different from zero, hence $y \in V$. We have now proved that $D \subseteq \{V\}_{A(f)}$. Now Lemma 1 shows that $A_v$ is a convex function algebra and Lemma 2 can be applied to show that $\{z\}_{A(f)}$ are connected in $M_{A(f)}$ for every $z \in V$. In particular $\{y\}_{A(f)}$ has no isolated points in $M_{A(f)}$. Since $D$ is an open subset of $M_{A(f)}$ we can find $x_1 \in D \cap \{y\}_{A(f)}$, such that $x_1 \neq x$. But now we get $f(x_i) = f(x) = 0$ and then $x$ and $x_i$ are not different points in $M_{A(f)}$, a contradiction.

**Definition 2.** A point $x \in M_A$ is stationary if $\{x\}_B = \{x\}$ for every function algebra $B$ such that $A \subset B \subset C(M_A)$.

**Theorem 2.** Let $A$ be a function algebra such that every point $x \in M_A$ has a fundamental neighborhood system $\{W\}$ such that each $bW$ consists of stationary points, then $A$ is a resistent function algebra.

**Proof.** Suppose that $B$ is a function algebra such that

$$A \subset B \subset C(M_A)$$

and assume that $D = M_B - M_A$ is not empty. Let $z \in D$ and choose $y \in M_A$ such that $f(z) = f(y)$ for $f \in A$. Choose an open neighborhood $V$ of $y$ in $M_A$ such that $bV$ consists of stationary points. Let $W$ be a closed $B$-convex neighborhood of $z$ in $M_B$ such that $W \subset D$. Now $\{V\}_B \cap W$ is open and closed in $W$. We apply Shilov's Idempotent Theorem to the function algebra $B_w$. Hence we find $\{f_n\} \in B$ such that $\lim |f_n - 1|_{W \cap \{V\}_B} = 0$ while $\lim |f_n|_{W - \{V\}_B} = 0$. Choose a minimal support $F$ of $z$ such that $F \subseteq bW$. It follows from the principle of minimal supports that $F \subseteq bW \cap \{V\}_B$. Now we let $V$ shrink to $y$ in $M_A$ and it follows that $z \in \text{Hull}_y(bW \cap \{y\}_B)$. This holds for every $z \in D \cap \{y\}_B$ when $W$ is a closed $B$-convex neighborhood of $z$ such that $W \subset D$. Now we choose a strong boundary point $x \in D \cap \{y\}_B$ of the function algebra $B_{\{y\}_B}$ to obtain a contradiction.
DEFINITION 4. A point $x \in M_A$ is locally regular if there exists a neighborhood $V$ of $x$ such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of $y$ and $f(x) = 1$.

THEOREM 3. A locally regular point is a stationary point.

Proof. Let $x \in M_A$ be a locally regular point. Let $B$ be a function algebra such that $A \subset B \subset C(M_A)$. Let $D = M_B - M_A$ and assume that $\{x\}_B \cap D$ is not empty. Let $V$ be an open neighborhood of $x$ in $M_A$ such that to every $y \in V - \{x\}$ there exists $f \in A$ with $f = 0$ in a neighborhood of $y$ and $f(x) = 1$. Let $z \in \{x\}_B \cap D$ and choose a closed neighborhood $W$ of $z$ in $M_B$ such that $W \subset D - \{V\}_B$. Let $F$ be a minimal support of $z$ such that $F \subset bW$. It follows now that $F \subset \{x\}_B$ holds. Hence $z \in \text{Hull}_B(bW \cap \{x\}_B)$ and we obtain a contradiction if we choose a suitable point $z \in D \cap \{x\}_B$. Hence $\{x\}_B \cap D$ must be empty and it follows that $x$ is a stationary point.

THEOREM 4. Let $A$ be a regular function algebra on a compact set $X$. Then every point $x \in X \cap M_A$ is a stationary point.

Proof. Let $x \in X \cap M_A$ and put $R(x) = \{y \in M_A |$ there exists $g \in A$ with $g = 0$ in a neighborhood of $y$ and $g(x) = 1\}$. We shall now prove that $R(x) = M_A - \{x\}$ and then it follows from Theorem 3 that $x$ is a stationary point. Let $y \in M_A - \{x\}$ and choose $g \in A$ such that $g(y) = 1$ and $g(x) = 0$. Let $V = \{z \in M_A | |g(z)| > 1/2\}$ and let $W = \{z \in X | |g(z)| \leq 1/2\}$. We choose $f \in A$ such that $f = 0$ on $X - W$ and $f(x) = 1$. If $z \in V$ we can choose a minimal support $F$ of $z$ such that $F \subset X$. Obviously $F \cap (X - W)$ is not empty and the principle of minimal supports implies that $f(z) = 0$. Hence $f = 0$ on $V$ and $f(x) = 1$, i.e., $y \in R(x)$.

THEOREM 5. Let $F$ be a closed subset of $M_A$ and let $f \in CM_A$ be such that $f$ is locally approximable in $A$ at every point in $M_A - F$. Then $M_{A(f)} - M_A \subset \{\text{Hull}_A(F)\}_{A(f)}$.

Proof. Let $D = M_{A(f)} - M_A$. Let $K = \text{Hull}_{A(f)}(bD)$ and let $C = A(f)_K$. We have $D \subset K = M_C$ and $bD$ contains the Shilov boundary of $C$. Let $x \in bD$ be a strong boundary point of $C$. Assume that $x \in M_A - F$. Choose a closed neighborhood $V$ of $x$ in $M_A$ such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_V = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K | |h(x)| \geq 1/2\} \subset V_{A(f)}$. Let

$$D_1 = \{x \in D | |h(x)| > 1/2\}.$$  

The topological boundary $bD_1$ of $D_1$ in $K$ is obviously contained in the set $T = \{x \in bD | |h(x)| \geq 1/2\} \cup \{x \in K | |h(x)| = 1/2\}$. Choose a point
Now the local maximum principle shows that we can find a minimal support \( F \) of \( x_1 \) in \( C \) such that \( F \subseteq T \). Since \( |h(x)| > 1/2 \) it follows that \( F \cap bD \) contains an open subset of \( F \). Since \( F \subseteq T \subseteq \{V\}_{A(f)} \) we have \( |g|_F \leq |g|_V \) for \( g \in A \). Now \( \lim |g_n - f|_{F \cap bD} \leq \lim |g_n - f|_V = 0 \) and the principle of minimal supports shows that \( \lim g_n(x) = f(x_1) \) holds. Now we also have \( x_1 \in \{y_i\}_{A(f)} \) for some point \( y_i \in V \). Hence \( f(y_i) = \lim g_n(y_i) = g_n(x) = f(x_1) \) and then \( x_1 \) and \( y_1 \) cannot be different points in \( M_{A(f)} \), a contradiction. We have now proved that every strong boundary point of \( C \) must belong to \( F \). It follows that \( S_c \subseteq F \) and hence \( M_{A(f)} - M_A \subseteq \text{Hull}_{A(f)}(F) \). This implies that \( M_{A(f)} - M_A \subseteq \{\text{Hull}_{A(f)}(F)\} \).

**Lemma 3.** Let \( A \) be a function algebra on a compact set \( X \). Let \( F \) be a closed subset of \( X \). Then there exists a point \( x \in F \) such that if \( m \) is a representing measure of \( x \) in \( A \) with \( m(F) = 1 \) then \( m = \delta_x \), i.e., \( m \) is the unit point mass at \( x \).

**Proof.** Choose a strong boundary point \( x \in F \) of the function algebra \( A_f \).

**Theorem 6.** Let \( A \subseteq B \subseteq C(M_A) \). Let \( f \in B \) be such that \( f \in H_0(A) \). Then \( f \) is constant on each fiber \( \{x\}_B \) for \( x \in M_A \).

**Proof.** If \( x \in M_B \) we denote by \( y(x) \) the point in \( M_A \) such that \( x \in \{y(x)\}_B \). Let \( d(x) = |f(x) - f(y(x))| \) and assume that \( d(x) \) is different from zero. Let \( F = \{x \in M_B \mid d(z) = \|d\| = \sup d(z)\} \). Obviously \( F \) is a closed subset of \( M_B \) and \( F \cap M_A \) is empty. Let \( x \in F \) and choose an open neighborhood \( V \) of \( y(x) \) in \( M_A \) such that there exists \( \{g_n\} \subseteq A \) with \( \lim |g_n - f|_V = 0 \). Choose now a closed neighborhood \( W \) of \( x \) in \( M_B \) such that \( W \subseteq \{V\}_{M_B} \cap (M_B - M_A) \). Let \( T \) be a minimal support of \( x \) such that \( T \subseteq bW \). Now we can find a positive measure on \( T \) such that \( g(z) = \int g dm \) from \( g \in B \). It follows that \( |f(x) - g_n(y(x))| = |f(x) - g_n(x)| \leq \int |f - g_n| dm \) for every \( n \). Hence we also get

\[
|f(x) - f(y(x))| \leq \int |f(z) - f(y(z))| dm(z) .
\]

It follows that \( |f(z) - f(y(z))| = \|d\| \) for every \( z \in T \), hence \( T \subseteq F \). We have now proved that \( x \in \text{Hull}_{A_f}(bW \cap F) \) for every \( x \in F \) and every closed neighborhood \( W \) of \( x \) such that \( W \subseteq (M_B - M_A) \). Now we derive a contradiction from Lemma 3.

**Theorem 7.** Let \( f \in C(M_A) \) and suppose that \( f \) is locally approximable in \( A \) at every point where \( f \) is different zero. Then \( M_{A(f)} = M_A \) and \( \text{Hull}_{A(f)}(F) = \text{Hull}_{A(f)}(F) \) for every closed subset \( F \) of \( M_A \).
Proof. Let $F$ be a closed subset of $M_A$ such that $F = \text{Hull}_{A(f)}(F)$. Let us put $G = \text{Hull}_{A(F)}(F)$ and assume that $D = G - F$ is not empty. Let $C = A(f)_G$. We see that the Shilov boundary $S_C$ of $C$ meets $D$. Hence we can find $x \in D$ such that $x$ is a strong boundary point of $C$. Let us assume that $f(x) \neq 0$. Choose a closed neighborhood $V \subset (M_A - F)$ of $x$ in $M_A$ such that there exist $\{g_n\} \in A$ with $|g_n - f|_V = 0$. Now we choose $h \in C$ such that if $P(h) = \{x \in G \mid h(x) = |h|_G\}$ then $x \in P(h)$ and $P(h) \subset V$ with $P(h) \cap bV$ empty. Since $h \in C$ we can find $\{h_n\} \in A$ with $|h_n - h|_{V \cap G} = 0$. Now the local maximum principle shows that $|g(x)| \leq |g|_{bV \cap G}$ for $g \in A$. It follows that $|h(x)| = \lim |h_n(x)| \leq \lim |h_n|_{bV \cap G} = |h|_{bV \cap G}$, contradiction to the fact that $P(h) \cap bV$ is empty. Hence we have proved that if $x \in D$ is a strong boundary point of $C$ then $f(x) = 0$. If $x \in D$ we can choose a minimal support $T$ of $x$ such that $T \subset S_C$. Since $F = \text{Hull}_{A(f)}(F)$ it follows that $T \cap D$ is not empty. Since $f = 0$ on $S_C \cap D$ it follows from the principle of minimal supports that $f(x) = 0$. Hence we have proved that $f = 0$ on $D$. But then $A(f)_D = A_D$ and it follows easily that $D$ cannot contain any strong boundary point of $C$. Hence $S_C \subset F$ which shows that $D$ must be empty. We have now proved that $\text{Hull}_{A(F)} = \text{Hull}_{A(f)}(F)$ for every closed subset $F$ of $M_A$. In particular we see that $Z(f) = \{x \in M_A \mid f(x) = 0\}$ is an $A$-convex set and using Theorem 5 it follows easily that $M_A = \text{Hull}_{A(f)}$.

Corollary 1. $M_A = M_{H(A)}$ and $\text{Hull}_{A(F)} = \text{Hull}_{H(A)}(F)$ for every closed subset $F$ of $M_A$.

Theorem 8. If $H(A)$ is a resistent function algebra then $A$ is a resistent function algebra.

Proof. If $A$ is not a resistent function algebra we can find $g_1, \ldots, g_k \in C(M_A)$ such that $g_1, \ldots, g_k$ have no common zero on $M_A$ while $g_i(z) = \cdots = g_k(z) = 0$ for some point $z \in M_{A(g_1, \ldots, g_k)}$. Because $H(A)$ is resistent we can find $h_1, \ldots, h_k$, where each $h_i$ is a polynomial in $g_1, \ldots, g_k$ with coefficients in $H_0(A)$, such that $|h_1g_1 + \cdots + h_kg_k - 1|_{H_A} < 1/2$. Let $h_i = \sum f_{i,v}g^v$, where $v$ runs over a finite set of multi-indices $(v_1, \ldots, v_k)$ and $g^v = g_1^{v_1} \cdots g_k^{v_k}$. Each $f_{i,v} \in H_0(A)$ and we define $f_{i,v}$ on $M_{A(g_1, \ldots, g_k)}$ by letting $f_{i,v}$ be constant on each fiber of $M_{A(g_1, \ldots, g_k)}$ over points of $M_A$. Each $g^v$ is defined on $M_{A(g_1, \ldots, g_k)}$ in the usual way. In this way we can extend each $h_i$ to $M_{A(g_1, \ldots, g_k)}$. Call these extensions $H_1, \ldots, H_k$. It is easily seen that $H = H_1g_1 + \cdots + H_kg_k$ is locally approximable in $A(g_1, \ldots, g_k)$ on $M_{A(g_1, \ldots, g_k)}$. Now $H(z) = 0$ while $|H - 1|_{H_A} < 1/2$ and since $M_A$ contains the Shilov boundary of $A(g_1, \ldots, g_k)$ we derive a contradiction from Corollary 1.

Theorem 9. Let $f \in C(M_A)$ be such that $f^n + a_1f^{n-1} + \cdots + a_n = 0$
on $\mathcal{M}_A$ where $a_1 \cdots a_n \in A$, then $\mathcal{M}_A = \mathcal{M}_{A(f)}$.

**Proof.** Let $g = n f^{n-1} + (n - 1) a_1 f^{n-2} + \cdots + a_n$. It is well known that $f$ is locally approximable in $A$ at every point $x \in \mathcal{M}_A$ where $g(x)$ is different from zero. (See [1], Th. 3.2.5, p. 71.) It follows that $g$ is locally approximable in $A$ at every point where $g$ is different from zero. Now Theorem 7 shows that $Z(g)$ is $A$-convex and then Theorem 5 shows that $\mathcal{M}_{A(f)} - \mathcal{M}_A \subset \{Z(g)\}_{A(f)}$. Let us put $B = A_{Z(g)}$, then $\mathcal{M}_B = Z(g)$ and the restriction of $f$ to $\mathcal{M}_B$ satisfies the equation $n f^{n-1} + (n - 1) b_1 f^{n-2} + \cdots + b_{n-1} = 0$ where $b_i \in B$ are the restrictions of $a_i$ to $Z(g)$. Since $\mathcal{M}_{A(f)} - \mathcal{M}_A \subset \{Z(g)\}_{A(f)}$ we see that $\mathcal{M}_{B(f)} - \mathcal{M}_B$ is not empty if $\mathcal{M}_{A(f)} - \mathcal{M}_A$ is not empty. Hence we can use induction over $n$ to prove that $\mathcal{M}_{A(f)} = \mathcal{M}_A$.

Let $A$ be a function algebra. If $F$ is a closed subset of $\mathcal{M}_A$ we have defined the function algebra $H(F)$. We are now interested in the maximal ideal space of $H(F)$.

**Definition.** If $F$ is a closed subset of $\mathcal{M}_A$ we put $\hat{F} = \{y \in \mathcal{M}_A | \{y\}_{H(F)} \cap \mathcal{M}_{H(F)}$ is not empty\}.

**Definition.** A natural set in $\mathcal{M}_A$ is a closed subset $F$ of $\mathcal{M}_A$ such that $F = \mathcal{M}_{H(F)}$.

**Lemma 4.** $(\cap F_a) \subset \cap \hat{F}_a$ for every family $\{F_a\}$ of closed subsets of $\mathcal{M}_A$.

**Proof.** Let $y \in \mathcal{M}_A$ be such that $y \in (\cap F_a)$. Hence there exists a complex-valued homomorphism $C$ of $H(\cap F_a)$ such that $C(g) = g(y)$ for $g \in A$. If $f \in H(F_a)$ the restriction of $f$ to $\cap F_a$ obviously gives an element of $H(\cap F_a)$. Hence $C$ can be restricted to $H(F_a)$ and we obtain a complex-valued homomorphism of $H(F_a)$ such that $C(g) = g(y)$ for $g \in A$.

**Theorem 10.** Let $F$ be a closed subset of $\mathcal{M}_A$ such that $F = \hat{F}$, then $\mathcal{M}_{H(F)} = F$.

**Proof.** Let $f \in H_a(F)$ and define $d(x) = |f(x) - f(y(x))|$ on $\mathcal{M}_{H(F)}$ where $y(x)$ is the point in $F$ such that $g(x) = g(y(x))$ for $g \in A$. Assume that $d$ is not identical zero. Let $D = \{x \in \mathcal{M}_{H(F)} | d(x) > 0\}$. Obviously $D \cap F$ is empty and hence $D$ lies off the Shilov boundary of $H(F)$. Hence $D \subset K = \text{Hull}_{H(F)}(bD)$. Let us put $C = H(F)_K$ and choose $x \in bD$ such that $x$ is a strong boundary point of $C$. Choose a closed neighborhood $V$ of $y(x)$ in $\mathcal{M}_A$ such that there exists $\{g_n\} \in A$ with $\lim |g_n - f|_{\cap F} = 0$. Now we choose $h \in C$ such that $h(x) = |h|_K = 1$ and $\{x \in K | |h(x)| \geq 1/2\} \subset \{V \cap F\}_{H(F)}$. Now we obtain a con-
tradiction using the same argument as in the final part of Theorem 5. Hence we have proved that if \( f \in H_\alpha(F) \) then \( f \) is constant on each fiber \( \{x\}_{H_\alpha(F)} \) when \( x \in F \). Since \( H_\alpha(F) \) is a dense subalgebra of \( H(F) \) it follows that \( F = M_{H_\alpha(F)} \).

**Corollary 2.** If \( \{F_a\} \) is a family of natural set of \( M_\Delta \) then \( \cap F_a \) is a natural set.

**Proof.** Lemma 4 shows that \( (\cap F_a)^\sim \subset \cap \hat{F}_a = \cap F_a \) and then Theorem 10 implies that \( \cap F_a \) is a natural set.

**Definition.** If \( F \) is a closed subset of \( M_\Delta \) then \( B(F) \) is the intersection of all natural sets containing \( F \). \( B(F) \) is called the barrier of \( F \).

Corollary 2 shows that \( B(F) \) is the smallest natural set containing a closed subset \( F \) of \( M_\Delta \).

**Lemma 5.** Let \( F \) be a natural set. Let \( f \in H(F) \) and let \( F_1 = \{x \in F \mid |f(x)| \leq 1\} \). Then \( F_1 \) is a natural set.

**Proof.** Let \( z \in M_{H(F)} \). If \( g \in H(F) \) the restriction of \( g \) to \( F_1 \) gives an element of \( H(F_1) \). It follows that \( g(z) = g(y) \) for some point \( y \in M_{H(F)} \) when \( g \in H(F) \). In particular \( f(z) = f(y) \) and since \( |f(z)| \leq |f| \) \( F_1 \) it follows that \( y \in F_1 \). Hence we have proved that \( F_1 = \hat{F}_1 \) and now Theorem 10 implies that \( F_1 \) is a natural set.

**Theorem 11.** Let \( F \) be a closed subset of \( M_\Delta \). Let \( S(F) \) be the Shilov boundary of \( H(B(F)) \). Then \( S(F) \subset F \).

**Proof.** Assume that \( S(F) \) meets \( B(F) - F \). Hence we can find \( x \in B(F) - F \) such that \( x \) is a strong boundary point of \( H(B(F)) \). Now we can choose \( f \in H(B(F)) \) such that \( F_1 = \{x \in B(F) \mid |f(x)| \leq 1\} \) contains \( F \) and omits the point \( x \).

Lemma 5 shows that \( F_1 \) is a natural set, a contradiction to the fact that \( B(F) \) is the smallest natural set containing \( F \).

We finally give some examples of natural subsets of \( M_\Delta \).

**Definition.** An \( A \)-analytic polyhedron \( P \) is a closed set in \( M_\Delta \) of the form \( P = \{x \in V \mid |f_\alpha(x)| \leq 1 \} \) where \( V \) is an open neighborhood of \( P \) and \( \{f_\alpha\} \) is a family in \( H_\alpha(V) \).

**Theorem 12.** An \( A \)-analytic polyhedron is a natural set.
Proof. Let $U$ be an open neighborhood of $P$ and $W$ a closed set containing $U$ such that $W \subset V$. Now we can find finitely many \{f_{a}\}$, say $f_{1} \cdots f_{k}$ such that $P_{i} = \{x \in W \mid f_{i}(x) \leq 1, \ i = 1 \cdots k\}$ is contained in $U$. Now we can prove that $P_{i}$ is a natural set using the same argument as in the final part of Theorem 5. Finally we let $U$ shrink to $P$ and obtain natural sets $\{P_{u}\}$ such that $P = \cap P_{u}$. Now Corollary 2 shows that $P$ is a natural set.

**Definition.** If $F$ is a closed subset of $M_{A}$ we put $R_{a}(F) = \{h \in C(F) \mid h = f/g \text{ where } f, g \in A \text{ and } g \text{ has no zero on } F\}$.

We let $R(F)$ be the function algebra on $F$ generated by $R_{a}(F)$.

**Definition.** If $F$ is a closed subset of $M_{A}$ we put $\text{Hull } R_{a}(F) = \{x \in M_{A} \mid g(x) \in g(F) \text{ for } g \in A\}$.

**Theorem 13.** $M_{R(F)} = \text{Hull } R(F)$ for every closed set $F$ in $M_{A}$ and if $M_{R(F)} = F$ then $F$ is a natural set.

**Proof.** If $y \in M_{R(F)}$ we choose $x \in M_{A}$ such that $g(y) = g(x)$ for $g \in A$. It is easily seen that $x \in \text{Hull } R(F)$ and that $(f/g)(y) = f(x)/g(x)$ when $f/g \in R_{a}(F)$. Since $R_{a}(F)$ is dense in $R(F)$ it follows that $y$ is uniquely determined by $x$. Conversely if we choose $x \in \text{Hull } R(F)$ then the mapping $X; f/g \rightarrow f(x)/g(x)$ is well defined on $R_{a}(F)$. We have $|f(x)/g(x)| \leq |f/g|_{F}$ for if $f(x) = g(x)$ while $|f/g|_{F} < 1$ we see that $(g - f)$ is different from zero on $F$ and hence $(g - f)(x) \in (g - f)(F)$ is different from zero, a contradiction. Hence we can extend $X$ to $R(F)$ and we obtain a complex-valued homomorphism on $R(F)$ such that $g$ is mapped into $g(x)$ when $g \in A$. This proves that $M_{R(F)} = \text{Hull } R(F)$. If $M_{R(F)} = F$ then Corollary 1 can be applied to prove that $F$ is a natural set.

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