

Pacific Journal of Mathematics

ON A PAPER OF RAO

MYRON GOLDSTEIN

ON A PAPER OF RAO

MYRON GOLDSTEIN

In this paper, we give an internal proof of Rao's theorem on meromorphic functions of bounded characteristic, i.e., a proof not using uniformization.

In addition, we discuss the classification theory of Riemann surfaces as it pertains to the class O_L of hyperbolic Riemann surfaces which admit no nonconstant Lindelöfian meromorphic functions. In particular, we show that $U_{HB} \subset O_L$ where U_{HB} denotes the class of hyperbolic Riemann surfaces on which there exist at least one bounded *MHB* minimal function.

We also show that there is no inclusion relation between O_L and O_{HD}^n , n a natural number, where O_{HD}^n denotes the class of hyperbolic Riemann surfaces for which the dimension of the vector lattice HD is at most n .

Finally, we generalize the *F.* and *M* Riesz theorem for H_1 of the unit disc to arbitrary open hyperbolic Riemann surfaces.

Let R be hyperbolic and R' be an arbitrary Riemann surface. The mapping $\varphi: R \rightarrow R'$ is called a Lindelöfian mapping, if for each $a' \in R'$,

$$G(z, a', \varphi) = \sum_{\varphi(a)=a'} n(a)g_R(z, a)$$

is convergent for $\varphi(z) \neq a'$ where $n(a)$ denotes the multiplicity of φ at a and $g_R(z, a)$ is the Green's function of R with pole at a . If R' is the Riemann sphere, then φ is called a Lindelöfian meromorphic function. Sario and Noshiro [6] have generalized the Nevanlinna theory to the class of meromorphic functions on an arbitrary Riemann surface and have shown that for hyperbolic surfaces, the meromorphic functions of bounded characteristic are precisely the Lindelöfian meromorphic functions. Furthermore, they have shown that a meromorphic function $\varphi(z)$ on a hyperbolic Riemann surface R has bounded characteristic if and only if for each complex number a'

$$\log |\varphi(z) - a'| = G(z, \infty, \varphi) - G(z, a', \varphi) + \varphi_{a'}(z) - \varphi'_{a'}(z)$$

where $\varphi_{a'}(z)$ and $\varphi'_{a'}(z)$ are positive harmonic functions on R .

Let us now turn to the theorem of Rao.

THEOREM 1. *If a nonconstant meromorphic function φ on a hyperbolic Riemann surface R has bounded characteristic, then there exists at most one complex number a' such that the difference between*

the quasibounded components of φ_a , and φ'_a , is constant.

As stated in the introduction, we now give an internal proof of Rao's theorem. For the proof, we need the following lemmas.

Let Δ_1 denote the set of points in the Martin boundary Δ of R where the Martin function k_b cannot be represented as the sum of two nonproportional potentials. In addition, let χ denote the canonical measure of 1 and f a continuous mapping of R into a compact space. Denote by I_b the class of open sets $G \subset R$ for which $(k_b)_{R-G}$, the infimum of the class of positive superharmonic functions on R which are quasi everywhere on $R - G$ no smaller than k_b , is a potential. Let $f^\wedge(b) = \bigcap_{G \in I_b} \overline{f(G)}$. If $f^\wedge(b)$ consists of a single point, we denote the point by $\hat{f}(b)$. We shall now establish the following result.

LEMMA 1. *If s is a singular positive harmonic function on R , then s is defined χ a.e. on Δ_1 and is 0 χ a.e.*

Proof. Since s is singular, its quasibounded component is 0. It follows from a result in [2] that s is defined χ a.e. on Δ_1 and that

$$0 = \int_{\Delta_1} \hat{s}(b)k_b d\chi(b).$$

Since \hat{s} and k_b are positive, it follows that $\hat{s} = 0$ χ a.e.

A different proof of this result can be found in [3]. In addition to Lemma 1, we will use the following, which are proved in [3].

LEMMA 2. *$G(z, a', \varphi)$ has the fine limit 0 χ a.e. on Δ_1*

LEMMA 3. *If φ is a nonconstant Lindelöfian meromorphic function on a hyperbolic Riemann surface R , then $\hat{\varphi}$ is defined χ a.e. on Δ_1 and the set of points $\hat{\varphi}(b)$ is a set of positive capacity.*

We are now able to prove Theorem 1.

Proof. Suppose the conclusion of the theorem is false. Then for some function φ of bounded characteristic on R and two complex numbers $a'_1 \neq a'_2$,

$$\log |\varphi(z) - a'_i| = G(z, \infty, \varphi) - G(z, a_i, \varphi) + k_i + s_i(z) - s'_i(z)$$

for $i = 1$ and 2 . s_i and s'_i are singular positive harmonic functions on R and k_i is a constant.

Since φ has bounded characteristic, it is Lindelöfian, and we deduce from Lemmas 1, 2, and 3 that

$$\log |\hat{\varphi}(b) - a'_i| = k_i, \quad i = 1, 2,$$

for almost all $b \in \mathcal{A}_1$.

Hence the points $\hat{\varphi}(b)$ lie on both circles

$$|\zeta - a'_i| = e^{ki}, \quad i = 1, 2, a'_1 \neq a'_2.$$

In view of the second part of Lemma 3, this is a contradiction and the theorem is proved.

2. We now turn to the class O_L . Rao [5] has shown that $O_{HB} \subset O_L$. We shall prove the much stronger result that $U_{HB} \subset O_L$. In addition, we shall prove a stronger form of Rao's Corollary 2.

Let φ be an analytic mapping from a Riemann surface R into a Riemann surface R' . φ is called a Fatou mapping if φ has a continuous extension to a mapping from the Wiener compactification of R into the Wiener compactification of R' .

It is shown in [2] that every Lindelöfian mapping is a Fatou mapping. It is also shown in [2] that if $R \in U_{HB}$, there exists no nonconstant Fatou mapping of R into a parabolic surface. As immediate consequences of these results, we obtain the following:

THEOREM 2. $U_{HB} \subset O_L$.

THEOREM 3. *If there exists a nonconstant Lindelöfian map φ from a hyperbolic Riemann surface R into a parabolic Riemann surface R' , then $R \notin U_{HB}$.*

3. Constantinescu and Cornea [2] have shown that if R is of class U_{HB} , then $R \in U_{HD}$ where U_{HD} denotes the class of Riemann surfaces on which there exist at least one bounded *MHD* minimal function. Punch out n pairwise disjoint discs F_1, \dots, F_n from R and reflect $R - \bar{F}_i$ about ∂F_i for $i = 1, \dots, n$. Weld the reflections to $R - \bigcup_{i=1}^n F_i$ and denote the resulting surface by W . If $R \in U_{HB}$, then $W \in U_{HB}$. Since R is also of class U_{HD} , it follows that the Royden harmonic boundary of W possesses at least $n + 1$ points of positive harmonic measure and hence that $W \notin O_{HD}^n$. But by Theorem 2, $W \in O_L$. Hence $O_L \not\subset O_{HD}^n$. Since $O_{HD}^n \not\subset O_L$, we have established the following result.

THEOREM 4. *There is no inclusion relation between O_{HD}^n and O_L .*

We remark that Theorem 4 contains the result of Rao that there is no inclusion relation between O_{HD} and O_L .

It would be interesting to know if Theorem 4 can be strengthened.

ed to read that there is no inclusion relation between U_{HD} and O_L . The author has investigated this question but has been unable to settle it.

4. Let us now turn to the class H_1 of analytic functions on an open hyperbolic Riemann surface R whose moduli possess a harmonic majorant. We shall prove the following result.

THEOREM 5. *If $f \in H_1$, then f is defined χ a.e. on Δ_1 and*

$$f = \int_{\Delta_1} f(b)k_b d\chi_b .$$

Proof. Since $f \in H_1$, it follows that the Ref and the Imf can be written as the difference of quasibounded harmonic functions. Hence

$$\text{Ref} = \int_{\Delta_1} \widehat{\text{Ref}}(b)k_b d\chi(b)$$

and

$$\text{Imf} = \int_{\Delta_1} \widehat{\text{Imf}}(b)k_b d\chi(b) .$$

Thus

$$f = \int_{\Delta_1} \widehat{f}(b)k_b d\chi(b) .$$

REFERENCES

1. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, Princeton, 1960.
2. C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer-Verlag, Berlin, 1963.
3. J. L. Doob, *Conformally invariant cluster value theory*, Illinois J. Math. **5** (1961), 521-549.
4. M. Heins, *Lindelöfian maps*, Ann. of Math. (2) **62** (1955), 418-446.
5. K. V. R. Rao, *Lindelöfian meromorphic functions*, Proc. Amer. Math. Soc. **15** (1964), 109-113.
6. L. Sario and K. Noshiro, *Value Distribution Theory*, D. Van Nostrand and Co., Princeton, 1966.

Received January 17, 1968.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California 90024.

Each author of each article receives 50 reprints free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$ 8.00; single issues, \$ 3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$ 4.00 per volume; single issues \$ 1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners of publishers and have no responsibility for its content or policies.

Charles A. Akemann, <i>Invariant subspaces of $C(G)$</i>	421
Dan Amir and Zvi Ziegler, <i>Generalized convexity cones and their duals</i>	425
Raymond Balbes, <i>On (J, M, m)-extensions of order sums of distributive lattices</i>	441
Jan-Erik Björk, <i>Extensions of the maximal ideal space of a function algebra</i>	453
Frank Castagna, <i>Sums of automorphisms of a primary abelian group</i>	463
Theodore Seio Chihara, <i>On determinate Hamburger moment problems</i>	475
Zeev Ditzian, <i>Convolution transforms whose inversion function has complex roots in a wide angle</i>	485
Myron Goldstein, <i>On a paper of Rao</i>	497
Velmer B. Headley and Charles Andrew Swanson, <i>Oscillation criteria for elliptic equations</i>	501
John Willard Heidel, <i>Qualitative behavior of solutions of a third order nonlinear differential equation</i>	507
Alan Carleton Hindmarsh, <i>Pick's conditions and analyticity</i>	527
Bruce Ansgar Jensen and Donald Wright Miller, <i>Commutative semigroups which are almost finite</i>	533
Lynn Clifford Kurtz and Don Harrell Tucker, <i>An extended form of the mean-ergodic theorem</i>	539
S. P. Lloyd, <i>Feller boundary induced by a transition operator</i>	547
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, <i>A new proof of the maximum principle for doubly-harmonic functions</i>	567
Robert Einsohn Mosher, <i>The product formula for the third obstruction</i>	573
Sam Bernard Nadler, Jr., <i>Sequences of contractions and fixed points</i>	579
Eric Albert Nordgren, <i>Invariant subspaces of a direct sum of weighted shifts</i>	587
Fred Richman, <i>Thin abelian p-groups</i>	599
Jordan Tobias Rosenbaum, <i>Simultaneous interpolation in H_2. II</i>	607
Charles Thomas Scarborough, <i>Minimal Urysohn spaces</i>	611
Malcolm Jay Sherman, <i>Disjoint invariant subspaces</i>	619
Joel John Westman, <i>Harmonic analysis on groupoids</i>	621
William Jennings Wickless, <i>Quasi-isomorphism and TFM rings</i>	633
Minoru Hasegawa, <i>Correction to "On the convergence of resolvents of operators"</i>	641