

# Pacific Journal of Mathematics

**PICK'S CONDITIONS AND ANALYTICITY**

ALAN CARLETON HINDMARSH

## PICK'S CONDITIONS AND ANALYTICITY

A. C. HINDMARSH

Let  $w(z)$  be a function in the open upper half plane (UHP) with values in UHP, and let  $P_n = (d_{ij})$  be the  $n \times n$  matrix of difference quotients

$$d_{ij} = \frac{w(z_i) - \overline{w(z_j)}}{z_i - \bar{z}_j}$$

formed from any  $n$  points  $z_1, z_2, \dots, z_n \in \text{UHP}$ . It was shown by G. Pick that if  $w(z)$  is also analytic in UHP, then the  $P_n$  are all nonnegative definite Hermitian matrices (denoted  $P_n \geq 0$ ). In what follows, two converse results are derived.

(1) If  $D$  is a domain in UHP,  $w(z)$  is continuous in  $D$  and has values in UHP, and  $P_3 \geq 0$  for all choices of the  $z_1, z_2, z_3 \in D$ , then  $w(z)$  is analytic in  $D$ . It is well known that the condition  $P_2 \geq 0$  does not imply anything of this sort, but corresponds only to a distance-shrinking property of  $w(z)$  in the noneuclidean geometry of UHP.

(2) If  $w$  is as before, but  $P_n \geq 0$  for all  $n$  and all  $z_1, \dots, z_n \in D$ , i.e.,  $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$  is a nonnegative definite kernel in  $D$ , then  $w(z)$  is analytic in  $D$  and has an analytic extension to UHP whose values are in UHP.

The central idea of result (1) is to consider the kernel  $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$  for  $z, \zeta$  in a neighborhood of a point  $z_0 \in D$  and to interpret the 3<sup>rd</sup> Pick condition  $P_3 \geq 0$  locally at  $z_0$ , thereby deriving coefficient inequalities for  $K$  at  $(z_0, z_0)$ . This idea is made explicit in the following lemma on general kernels:

LEMMA. Let  $D$  be an open set in  $R^n$ , and let

$$K(u, v) = K(u_1, \dots, u_n; v_1, \dots, v_n)$$

be a  $C^2$  kernel defined for  $u, v \in D$ , with  $K(u, v) = \overline{K(v, u)}$ . If  $K \geq 0$  of order  $n + 1$  in  $D$ , i.e.,  $(k_{ij}) \geq 0$  for the  $(n + 1) \times (n + 1)$  matrix with elements  $k_{ij} = K(u^i, u^j)$  formed from any  $n + 1$  points  $u^0, u^1, \dots, u^n \in D$ , then for each  $u \in D$  we have

$$M(u) = \begin{pmatrix} K & K_{v_j} \\ K_{u_i} & K_{u_i v_j} \end{pmatrix} \Big|_{(u, u)} \geq 0.$$

Here  $K_{v_j}$  refers to the row vector  $(K_{v_1} K_{v_2} \dots K_{v_n})$ ,  $K_{u_i}$  to a similar column vector, and  $K_{u_i v_j}$  to an  $n \times n$  matrix. Subscripts on  $K$  denote partial differentiation.

*Proof.* Fix  $u \in D$ . For small positive  $h$ , let  $u^i = (u_1^i, \dots, u_n^i)$ , where  $u_k^i = \begin{cases} h & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$ . Then let  $K(h)$  be the  $(n + 1) \times (n + 1)$  matrix  $(k_{ij})$ ,  $0 \leq i, j \leq n$ ,  $k_{ij} = K(u + u^i, u + u^j)$ . For all small  $h$ ,  $K(h) \geq 0$ . Now form  $\tilde{K}(h) = (\tilde{k}_{ij})$  where

$$\tilde{k}_{00} = k_{00}, \tilde{k}_{0j} = \frac{k_{0j} - k_{00}}{h}, \tilde{k}_{i0} = \frac{k_{i0} - k_{00}}{h}, \tilde{k}_{ij} = \frac{k_{ij} + k_{00} - k_{0j} - k_{i0}}{h^2} \quad (i, j \geq 1).$$

If  $K, K_{u_i}$ , etc., denote the value and various derivatives of  $K$  at  $(u, u)$ , then we have

$$\begin{aligned} k_{00} &= K, \quad k_{0j} = K + hK_{v_j} + \frac{h^2}{2}K_{v_jv_j} + o(h^2), \\ k_{i0} &= K + hK_{u_i} + \frac{h^2}{2}K_{u_iu_i} + o(h^2), \\ k_{ij} &= K + h(K_{u_i} + K_{v_j}) + \frac{h^2}{2}(K_{u_iu_i} + 2K_{u_iv_j} + K_{v_jv_j}) + o(h^2), \end{aligned}$$

and so, as  $h \rightarrow 0$ ,

$$\tilde{k}_{00} = K, \tilde{k}_{0j} = K_{v_j} + o(1), \tilde{k}_{i0} = K_{u_i} + o(1), \tilde{k}_{ij} = K_{u_iv_j} + o(1) \quad (i, j, \geq 1).$$

But  $K(h) \geq 0 \Leftrightarrow \tilde{K}(h) \geq 0$ , because the change  $K \rightarrow \tilde{K}$  in the associated quadratic form corresponds to the invertible linear change of coordinates in  $C^{n+1}$  given by  $X_0 = \tilde{X}_0 - (\sum_1^n \tilde{X}_i)/h, X_i = \tilde{X}_i/h$  ( $i \geq 1$ ). Hence we conclude that  $\lim_{h \rightarrow 0} \tilde{K}(h) = M(u) \geq 0$ .

We wish to apply the lemma to the case of a kernel  $K(z, \zeta)$  defined for  $z, \zeta \in D, D$  being an open set in the plane, with  $K \in C^2$ , and  $K(z, \zeta) = \overline{K(\zeta, z)}$ . If we have  $K \geq 0$  of order 3 in  $D$ , i.e.,  $(K(z_i, z_j)) \geq 0$  for the  $3 \times 3$  matrix formed from  $z_1, z_2, z_3 \in D$ , we deduce that

$$N(z) = \begin{pmatrix} K & K_\xi & K_\eta \\ K_x & K_{x\xi} & K_{x\eta} \\ K_y & K_{y\xi} & K_{y\eta} \end{pmatrix} \Big|_{(z,z)} \geq 0 \quad (z = x + iy, \zeta = \xi + i\eta)$$

for  $z \in D$ , by applying the lemma to  $J(u, v) = K(u_1 + iu_2, v_1 + iv_2)$  with  $n = 2$ . Further, by a change of coordinates given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & i/2 \end{pmatrix},$$

we obtain

$$AN(z)A^* = M(z) = \begin{pmatrix} K & K_{\bar{z}} & K_{\zeta} \\ K_z & K_{z\bar{z}} & K_{z\zeta} \\ K_{\bar{z}} & K_{z\bar{z}} & K_{z\zeta} \end{pmatrix} \Big|_{(z,z)} \geq 0.$$

To apply this last result to the present problem, let  $D$  be an open set in UHP, let  $w(z)$  be given in  $D$  with values in UHP and with  $P_3 \geq 0$  in  $D$ , and suppose first that  $w \in C^2$ . Then  $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$  is an admissible kernel, and we are led to the  $3 \times 3$  coefficient matrix  $M(z) = (m_{ij}) \geq 0$ . Putting  $A = z - \bar{\zeta}$ ,  $B = w(z) - \overline{w(\zeta)}$ , the required derivatives of  $K = B/A$  at  $(z, \zeta)$  are

$$K_{\zeta} = \frac{AB_{\zeta} - A_{\zeta}B}{A^2} = -\frac{\overline{w_z(\zeta)}}{A}, \quad K_z = \frac{w_z(z)}{A}, \quad K_{z\zeta} = \frac{\overline{w_z(\zeta)}}{A^2},$$

$$K_{z\bar{z}} = 0, \quad \text{etc.}$$

But  $M(z) \geq 0$  implies in particular that

$$0 \leq m_{22}m_{33} - |m_{23}|^2 = K_{z\bar{z}}K_{z\zeta} - |K_{z\zeta}|^2 \Big|_{(z,z)} = -|K_{z\zeta}(z, z)|^2.$$

Hence  $K_{z\zeta}(z, z) = 0$ , and so  $w_z(z) = 0$ . I.e., the Cauchy-Riemann Equations hold in  $D$ , and  $w(z)$  is analytic in  $D$ .

In order to remove the assumption  $w \in C^2$ , we use a standard mollification argument. In a neighborhood of  $z_0 \in D$ , we approximate the continuous function  $w(z)$  by mollified functions  $w_{\delta}(z)$ , such that  $w_{\delta} \in C^2$  and  $w_{\delta} \rightarrow w$  uniformly in a neighborhood of  $z_0$ . Since the property  $P_3 \geq 0$  is additive and positive-homogeneous in  $w$ , we see also that  $P_3 \geq 0$  for each  $w_{\delta}$  as well as for  $w$ . We therefore know that  $w_{\delta}$  is analytic in a neighborhood of  $z_0$ . By uniform convergence, so is  $w$ . Since  $z_0$  was arbitrary,  $w(z)$  is analytic throughout  $D$ .

From the above proof, it is clear that the hypotheses in statement (1) are considerably stronger than they need be. First, the fact that only  $m_{22}m_{33} - |m_{23}|^2 \geq 0$  was used means that  $P_3 = (k_{ij})$  need only be nonnegative definite on the subspace  $L_3 = \{(X_i) \in C^3 : \sum X_i = 0\}$  of complex dimension 2. For, in the notation of the proof of the lemma, the latter condition is equivalent to

$$\begin{pmatrix} \tilde{k}_{11} & \tilde{k}_{12} \\ \tilde{k}_{21} & \tilde{k}_{22} \end{pmatrix} \geq 0.$$

The analogous form of the lemma, in which  $(K(u^i, u^j)) \geq 0$  on  $L_{n+1}$  for  $u^0, u^1, \dots, u^n \in D = (K_{u^i v^j}(u, u)) \geq 0$ , is similarly proved. Secondly, there is now no need for the values of  $w(z)$  to lie in UHP. These two alterations mean that the analyticity result holds when  $w(z)$  is a continuous "infinitesimal transformation" of the class of maps of

$D$  satisfying  $P_3 \geq 0$ , i.e.,  $w(z) = \partial f_t(z)/\partial t|_{t=0}$ , where  $f_t, 0 \leq t \leq t_0$ , is a family of functions in  $D$  satisfying  $P_3 \geq 0$  in  $D$  for all  $t$ , and  $f_0(z) = z$ . The class of such  $w(z)$  is in fact characterized by the condition  $P_3 \geq 0$  on  $L_3$  (and likewise for general  $n$ ). The positivity hypothesis could also be weakened from a global condition to a local one, but since  $D$  is arbitrary and analyticity is a local property, this would be a trivial alteration. To summarize, we state:

**THEOREM 1.** *Let  $w(z)$  be a continuous function in an open subset  $D$  of UHP. If, for all  $z_1, z_2, z_3 \in D$ , the  $3 \times 3$  matrix of difference quotients  $d_{ij} = \{w(z_i) - \overline{w(z_j)}\}/(z_i - \bar{z}_j)$  satisfies  $(d_{ij}) \geq 0$  on the subspace  $\{(X_i) \in C^3 : \sum X_i = 0\}$ , then  $w(z)$  is analytic in  $D$ .*

It should be noted here that result (1), in the weaker form, can also be easily proven from Pick's Theorem (below). However, the latter requires a proof that considerably more involved than that given here for Theorem 1.

The statement (2) gives a characterization of the class  $P$  of "positive" functions, analytic in UHP with values in UHP. It says that all of Pick's conditions together imply that  $w$  is the restriction to  $D$  of a  $P$  function. The proof depends on the following:

**PICK'S THEOREM.** *If  $z_1, \dots, z_n, w_1, \dots, w_n \in \text{UHP}$  and  $P_n = (d_{ij}) \geq 0$  for the  $n \times n$  matrix of difference quotients  $d_{ij} = (w_i - \bar{w}_j)/(z_i - \bar{z}_j)$ , then there is a function  $f \in P$  for which  $f(z_i) = w_i$  for  $1 \leq i \leq n$ .*

Now if  $w(z)$  is continuous in  $D$  and  $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$  is nonnegative definite (of infinite order) in  $D$ , we can choose a dense sequence  $(z_i)$  from  $D$  and apply Pick's Theorem for each  $n$ . Because  $P$  is a normal family, the  $P$  functions so gotten have a normally convergent subsequence, and the analytic limit agrees with  $w$  in  $D$ . We thus obtain

**THEOREM 2.** *Let  $w(z)$  be a continuous function in a domain  $D \subset \text{UHP}$  with values in UHP. If  $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$  is a non-negative definite kernel in  $D$ , then  $w$  is analytic in  $D$  and has an analytic extension to UHP whose values are in UHP.*

I wish to take this opportunity to express my deep gratitude for Prof. Loewner's guidance and my sorrow at his loss.

I wish to take this opportunity to express my sorrow at the loss of Professor Charles Loewner, who, as my thesis advisor, inspired the work represented in this paper.

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