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Let $w(z)$ be a function in the open upper half plane (UHP) with values in UHP, and let $P_n = (d_{ij})$ be the $n \times n$ matrix of difference quotients

$$d_{ij} = \frac{w(z_i) - \overline{w(z_j)}}{z_i - \bar{z}_j}$$

formed from any n points $z_1, z_2, \cdots, z_n \in$ UHP. It was shown by G. Pick that if $w(z)$ is also analytic in UHP, then the P_n are all nonnegative definite Hermitian matrices (denoted $P_n \geqq 0$). In what follows, two converse results are derived.

(1) If D is a domain in UHP, $w(z)$ is continuous in D and has values in UHP, and $P_3 \geqq 0$ for all choices of the $z_1, z_2, z_3 \in D$, then $w(z)$ is analytic in D. It is well known that the condition $P_2 \geqq 0$ does not imply anything of this sort, but corresponds only to a distance-shrinking property of $w(z)$ in the noneuclidean geometry of UHP.

(2) If w is as before, but $P_n \geqq 0$ for all n and all $z_1, \cdots, z_n \in D$, i.e., $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is a nonnegative definite kernel in D, then $w(z)$ is analytic in D and has an analytic extension to UHP whose values are in UHP.

The central idea of result (1) is to consider the kernel $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ for z, ζ in a neighborhood of a point $z_0 \in D$ and to interpret the 3rd Pick condition $P_3 \geqq 0$ locally at z_0, thereby deriving coefficient inequalities for K at (z_0, z_0). This idea is made explicit in the following lemma on general kernels:

LEMMA. *Let D be an open set in $\boldsymbol{R}^n$, and let*

$$K(u, v) = K(u_1, \cdots, u_n; v_1, \cdots, v_n)$$

be a C^2 kernel defined for $u, v \in D$, with $K(u, v) = \overline{K(v, u)}$. If $K \geqq 0$ of order $n + 1$ in D, i.e., $(k_{ij}) \geqq 0$ for the $(n + 1) \times (n + 1)$ matrix with elements $k_{ij} = K(u^i, u^j)$ formed from any $n + 1$ points $u^0, u^1, \cdots, u^n \in D$, then for each $u \in D$ we have

$$M(u) = \begin{pmatrix} K & K_{v_j} \\ K_{u_i} & K_{u_i v_j} \end{pmatrix}\Bigg|_{(u,u)} \geqq 0 \, .$$

Here K_{v_j} refers to the row vector $(K_{v_1} K_{v_2} \cdots K_{v_n})$, K_{u_i} to a similar column vector, and $K_{u_i v_j}$ to an $n \times n$ matrix. Subscripts on K denote partial differentiation.

Proof. Fix $u \in D$. For small positive h, let $u^i = (u_1^i, \cdots, u_n^i)$, where $u_k^i = \begin{Bmatrix} h & \text{if} \quad k = i \\ 0 & \text{otherwise} \end{Bmatrix}$. Then let $K(h)$ be the $(n+1) \times (n+1)$ matrix (k_{ij}), $0 \leqq i, j \leqq n$, $k_{ij} = K(u + u^i, u + u^j)$. For all small h, $K(h) \geqq 0$. Now form $\tilde{K}(h) = (\tilde{k}_{ij})$ where

$$\tilde{k}_{00} = k_{00},\ \tilde{k}_{0j} = \frac{k_{0j} - k_{00}}{h},\ \tilde{k}_{i0} = \frac{k_{i0} - k_{00}}{h},\ \tilde{k}_{ij} = \frac{k_{ij} + k_{00} - k_{0j} - k_{i0}}{h^2} \qquad (i, j \geqq 1).$$

If K, K_{u_i}, etc., denote the value and various derivatives of K at (u, u), then we have

$$k_{00} = K,\ k_{0j} = K + hK_{v_j} + \frac{h^2}{2}K_{v_jv_j} + o(h^2)\ ,$$

$$k_{i0} = K + hK_{u_i} + \frac{h^2}{2}K_{u_iu_i} + o(h^2)\ ,$$

$$k_{ij} = K + h(K_{u_i} + K_{v_j}) + \frac{h^2}{2}(K_{u_iu_i} + 2K_{u_iv_j} + K_{v_jv_j}) + o(h^2)\ ,$$

and so, as $h \rightarrow 0$,

$$\tilde{k}_{00} = K,\ \tilde{k}_{0_j} = K_{v_j} + o(1),\ \tilde{k}_{i0} = K_{u_i} + o(1),\ \tilde{k}_{ij} = K_{u_iv_j} + o(1) \qquad (i, j, \geqq 1)\ .$$

But $K(h) \geqq 0 \Leftrightarrow \tilde{K}(h) \geqq 0$, because the change $K \rightarrow \tilde{K}$ in the associated quadratic form corresponds to the invertible linear change of coordinates in C^{n+1} given by $X_0 = \tilde{X}_0 - (\sum_1^n \tilde{X}_i)/h$, $X_i = \tilde{X}_i/h$ $(i \geqq 1)$. Hence we conclude that $\lim_{h \to 0} \tilde{K}(h) = M(u) \geqq 0$.

We wish to apply the lemma to the case of a kernel $K(z, \zeta)$ defined for $z, \zeta \in D$, D being an open set in the plane, with $K \in C^2$, and $K(z, \zeta) = \overline{K(\zeta, z)}$. If we have $K \geqq 0$ of order 3 in D, i.e., $(K(z_i, z_j)) \geqq 0$ for the 3×3 matrix formed from $z_1, z_2, z_3 \in D$, we deduce that

$$N(z) = \begin{pmatrix} K & K_\xi & K_\eta \\ K_x & K_{x\xi} & K_{x\eta} \\ K_y & K_{y\xi} & K_{y\eta} \end{pmatrix}\Bigg|_{(z,z)} \geqq 0 \qquad (z = x + iy, \zeta = \xi + i\eta)$$

for $z \in D$, by applying the lemma to $J(u, \mathrm{v}) = K(u_1 + iu_2, v_1 + iv_2)$ with $n = 2$. Further, by a change of coordinates given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & i/2 \end{pmatrix},$$

we obtain

$$AN(z)A^* = M(z) = \begin{pmatrix} K & K_{\bar\zeta} & K_\zeta \\ K_z & K_{z\bar\zeta} & K_{z\zeta} \\ K_{\bar z} & K_{\bar z\bar\zeta} & K_{\bar z\zeta} \end{pmatrix}\Bigg|_{(z,z)} \geqq 0 .$$

To apply this last result to the present problem, let D be an open set in UHP, let $w(z)$ be given in D with values in UHP and with $P_3 \geqq 0$ in D, and suppose first that $w \in C^2$. Then $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar\zeta)$ is an admissible kernel, and we are led to the 3×3 coefficient matrix $M(z) = (m_{ij}) \geqq 0$. Putting $A = z - \bar\zeta$, $B = w(z) - \overline{w(\zeta)}$, the required derivatives of $K = B/A$ at (z, ζ) are

$$K_\zeta = \frac{AB_\zeta - A_\zeta B}{A^2} = -\frac{\overline{w_{\bar z}(\zeta)}}{A}, \quad K_{\bar z} = \frac{w_{\bar z}(z)}{A}, \quad K_{z\zeta} = \frac{\overline{w_{\bar z}(\zeta)}}{A^2},$$

$$K_{\bar z\zeta} = 0, \qquad \text{etc.}$$

But $M(z) \geqq 0$ implies in particular that

$$0 \leqq m_{22}m_{33} - |m_{23}|^2 = K_{z\bar\zeta}K_{\bar z\zeta} - |K_{z\zeta}|^2|_{(z,z)} = -|K_{z\zeta}(z, z)|^2 .$$

Hence $K_{z\zeta}(z, z) = 0$, and so $w_{\bar z}(z) = 0$. I.e., the Cauchy-Riemann Equations hold in D, and $w(z)$ is analytic in D.

In order to remove the assumption $w \in C^2$, we use a standard mollification argument. In a neighborhood of $z_0 \in D$, we approximate the continuous function $w(z)$ by mollified functions $w_\delta(z)$, such that $w_\delta \in C^2$ and $w_\delta \to w$ uniformly in a neighborhood of z_0. Since the property $P_3 \geqq 0$ is additive and positive-homogeneous in w, we see also that $P_3 \geqq 0$ for each w_δ as well as for w. We therefore know that w_δ is analytic in a neighborhood of z_0. By uniform convergence, so is w. Since z_0 was arbitrary, $w(z)$ is analytic throughout D.

From the above proof, it is clear that the hypotheses in statement (1) are considerably stronger than they need be. First, the fact that only $m_{22}m_{33} - |m_{23}|^2 \geqq 0$ was used means that $P_3 = (k_{ij})$ need only be nonnegative definite on the subspace $L_3 = \{(X_i) \in C^3 : \sum X_i = 0\}$ of complex dimension 2. For, in the notation of the proof of the lemma, the latter condition is equivalent to

$$\begin{pmatrix} \tilde k_{11} & \tilde k_{12} \\ \tilde k_{21} & \tilde k_{22} \end{pmatrix} \geqq 0 .$$

The analogous form of the lemma, in which $(K(u^i, u^j)) \geqq 0$ on L_{n+1} for $u^0, u^1, \cdots, u^n \in D \Rightarrow (K_{u_i v_j}(u, u)) \geqq 0$, is similarly proved. Secondly, there is now no need for the values of $w(z)$ to lie in UHP. These two alterations mean that the analyticity result holds when $w(z)$ is a continuous "infinitesimal transformation" of the class of maps of

D satisfying $P_3 \geqq 0$, i.e., $w(z) = \partial f_t(z)/\partial t|_{t=0}$, where f_t, $0 \leqq t \leqq t_0$, is a family of functions in D satisfying $P_3 \geqq 0$ in D for all t, and $f_0(z) = z$. The class of such $w(z)$ is in fact characterized by the condition $P_3 \geqq 0$ on L_3 (and likewise for general n). The positivity hypothesis could also be weakened from a global condition to a local one, but since D is arbitrary and analyticity is a local property, this would be a trivial alteration. To summarize, we state:

THEOREM 1. *Let $w(z)$ be a continuous function in an open subset D of* UHP. *If, for all $z_1, z_2, z_3 \in D$, the 3×3 matrix of difference quotients $d_{ij} = \{w(z_i) - \overline{w(z_j)}\}/(z_i - \bar{z}_j)$ satisfies $(d_{ij}) \geqq 0$ on the subspace $\{(X_i) \in C^3 : \sum X_i = 0\}$, then $w(z)$ is analytic in D.*

It should be noted here that result (1), in the weaker form, can also be easily proven from Pick's Theorem (below). However, the latter requires a proof that considerably more involved than that given here for Theorem 1.

The statement (2) gives a characterization of the class P of "positive" functions, analytic in UHP with values in UHP. It says that all of Pick's conditions together imply that w is the restriction to D of a P function. The proof depends on the following:

PICK'S THEOREM. *If $z_1, \cdots, z_n, w_1, \cdots, w_n \in$ UHP and $P_n = (d_{ij}) \geqq 0$ for the $n \times n$ matrix of difference quotients $d_{ij} = (w_i - \bar{w}_j)/(z_i - \bar{z}_j)$, then there is a function $f \in P$ for which $f(z_i) = w_i$ for $1 \leqq i \leqq n$.*

Now if $w(z)$ is continuous in D and $K(z, \zeta) = \{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is nonnegative definite (of infinite order) in D, we can choose a dense sequence (z_i) from D and apply Pick's Theorem for each n. Because P is a normal family, the P functions so gotten have a normally convergent subsequence, and the analytic limit agrees with w in D. We thus obtain

THEOREM 2. *Let $w(z)$ be a continuous function in a domain $D \subset$ UHP with values in UHP. If $\{w(z) - \overline{w(\zeta)}\}/(z - \bar{\zeta})$ is a nonnegative definite kernel in D, then w is analytic in D and has an analytic extension to UHP whose values are in UHP.*

I wish to take this opportunity to express my deep gratitude for Prof. Loewner's guidance and my sorrow at his loss.

I wish to take this opportunity to express my sorrow at the loss of Professor Charles Loewner, who, as my thesis advisor, inspired the work represented in this paper.

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