PICK’S CONDITIONS AND ANALYTICITY

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Let \( w(z) \) be a function in the open upper half plane (UHP) with values in UHP, and let \( P_n = (d_{ij}) \) be the \( n \times n \) matrix of difference quotients

\[
d_{ij} = \frac{w(z_i) - w(z_j)}{z_i - \bar{z}_j}
\]

formed from any \( n \) points \( z_1, z_2, \ldots, z_n \in \text{UHP} \). It was shown by G. Pick that if \( w(z) \) is also analytic in UHP, then the \( P_n \) are all nonnegative definite Hermitian matrices (denoted \( P_n \geq 0 \)). In what follows, two converse results are derived.

1. If \( D \) is a domain in UHP, \( w(z) \) is continuous in \( D \) and has values in UHP, and \( P_3 \geq 0 \) for all choices of the \( z_1, z_2, z_3 \in D \), then \( w(z) \) is analytic in \( D \). It is well known that the condition \( P_2 \geq 0 \) does not imply anything of this sort, but corresponds only to a distance-shrinking property of \( w(z) \) in the noneuclidean geometry of UHP.

2. If \( w \) is as before, but \( P_n \geq 0 \) for all \( n \) and all \( z_1, \ldots, z_n \in D \), i.e., \( (w(z) - w(\zeta))/(z - \bar{\zeta}) \) is a nonnegative definite kernel in \( D \), then \( w(z) \) is analytic in \( D \) and has an analytic extension to UHP whose values are in UHP.

The central idea of result (1) is to consider the kernel \( K(z, \zeta) = (w(z) - w(\zeta))/(z - \bar{\zeta}) \) for \( z, \zeta \) in a neighborhood of a point \( z_0 \in D \) and to interpret the 3rd Pick condition \( P_3 \geq 0 \) locally at \( z_0 \), thereby deriving coefficient inequalities for \( K \) at \( (z_0, z_0) \). This idea is made explicit in the following lemma on general kernels:

**Lemma.** Let \( D \) be an open set in \( \mathbb{R}^n \), and let

\[
K(u, v) = K(u_1, \ldots, u_n; v_1, \ldots, v_n)
\]

be a \( C^2 \) kernel defined for \( u, v \in D \), with \( K(u, v) = \overline{K(v, u)} \). If \( K \geq 0 \) of order \( n + 1 \) in \( D \), i.e., \( (k_{ij}) \geq 0 \) for the \((n + 1) \times (n + 1)\) matrix with elements \( k_{ij} = K(u^i, u^j) \) formed from any \( n + 1 \) points \( u^0, u^1, \ldots, u^n \in D \), then for each \( u \in D \) we have

\[
M(u) = \begin{pmatrix} K & K_{v_j} \\ K_{u_i} & K_{u_i v_j} \end{pmatrix}_{(u, u)} \geq 0.
\]

Here \( K_{v_j} \) refers to the row vector \( (K_{v_1}, K_{v_2}, \ldots, K_{v_n}) \), \( K_{u_i} \) to a similar column vector, and \( K_{u_i v_j} \) to an \( n \times n \) matrix. Subscripts on \( K \) denote partial differentiation.
Proof. Fix $u \in D$. For small positive $h$, let $u^* = (u^i, \ldots, u^n)$, where $u^i_k = \begin{cases} h & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$. Then let $K(h)$ be the $(n + 1) \times (n + 1)$ matrix $(k_{ij})$, $0 \leq i, j \leq n$, $k_{ij} = K(u + u^i, u + u^j)$. For all small $h$, $K(h) \succeq 0$. Now form $\tilde{K}(h) = (\tilde{k}_{ij})$ where

$$
\tilde{k}_{00} = k_{00}, \quad \tilde{k}_{0j} = \frac{k_{0j} - k_{00}}{h}, \quad \tilde{k}_{i0} = \frac{k_{i0} - k_{00}}{h}, \quad \tilde{k}_{ij} = \frac{k_{ij} + k_{00} - k_{0j} - k_{i0}}{h^2} \\
(i, j \geq 1).
$$

If $K$, $K_u$, etc., denote the value and various derivatives of $K$ at $(u, u)$, then we have

$$
k_{00} = K, \quad k_{ij} = K + hK_{x^j} + \frac{h^2}{2}K_{x^j y^j} + o(h^2),
$$

$$
k_{i0} = K + hK_{u^i} + \frac{h^2}{2}K_{u^i y^i} + o(h^2),
$$

$$
k_{ij} = K + h(K_{u^i} + K_{u^j}) + \frac{h^2}{2}(K_{u^i u^i} + 2K_{u^i u^j} + K_{u^j u^j}) + o(h^2),
$$

and so, as $h \to 0$,

$$
\tilde{k}_{00} = K, \quad \tilde{k}_{0j} = K_{x^j} + o(1), \quad \tilde{k}_{i0} = K_{u^i} + o(1), \quad \tilde{k}_{ij} = K_{u^i u^j} + o(1) \\
(i, j \geq 1).
$$

But $K(h) \succeq 0 \iff \tilde{K}(h) \succeq 0$, because the change $K \to \tilde{K}$ in the associated quadratic form corresponds to the invertible linear change of coordinates in $C^{n+1}$ given by $X_0 = \tilde{X}_0 - (\sum_{i=1}^n \tilde{X}_i)/h, X_i = \tilde{X}_i/h$ $(i \geq 1)$. Hence we conclude that $\lim_{h \to 0} \tilde{K}(h) = M(u) \succeq 0$.

We wish to apply the lemma to the case of a kernel $K(z, \zeta)$ defined for $z, \zeta \in D, D$ being an open set in the plane, with $K \in C^3$, and $K(z, \zeta) = \overline{K(\zeta, z)}$. If we have $K \succeq 0$ of order 3 in $D$, i.e., $(K(z_i, z_j)) \succeq 0$ for the $3 \times 3$ matrix formed from $z_i, z_2, z_3 \in D$, we deduce that

$$
N(z) = \begin{pmatrix}
K & K_x & K_{x^2} \\
K_x & K_{x^2} & K_{x^3} \\
K_{x^2} & K_{x^3} & K_{x^4}
\end{pmatrix}
(z = x + iy, \zeta = \xi + i\eta) \succeq 0
$$

for $z \in D$, by applying the lemma to $J(u, v) = K(u_1 + iu_2, v_1 + iv_2)$ with $n = 2$. Further, by a change of coordinates given by the matrix

$$
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/2 & -i/2 \\
0 & 1/2 & i/2
\end{pmatrix},
$$
we obtain

$$AN(z)A^* = M(z) = \begin{pmatrix} K & K_\zeta & K_\zeta \\ K_\zeta & K_{z\zeta} & K_{z\zeta} \\ K_\zeta & K_{z\zeta} & K_{z\zeta} \end{pmatrix},$$

To apply this last result to the present problem, let $D$ be an open set in UHP, let $w(z)$ be given in $D$ with values in UHP and with $P_3 \geq 0$ in $D$, and suppose first that $w \in C^2$. Then $K(z, \zeta) = \{w(z) - w(\zeta)\}/(z - \zeta)$ is an admissible kernel, and we are led to the $3 \times 3$ coefficient matrix $M(z) = (m_{ij}) \geq 0$. Putting $A = z - \zeta, B = w(z) - w(\zeta)$, the required derivatives of $K = B/A$ at $(z, \zeta)$ are

$$K_\zeta = \frac{AB_\zeta - A_\zeta B}{A^2}, \quad K_z = \frac{w_z(z)}{A}, \quad K_{z\zeta} = \frac{w_z(\zeta)}{A^2},$$

$$K_{z\zeta} = 0, \quad \text{etc.}$$

But $M(z) \geq 0$ implies in particular that

$$0 \leq m_{22}m_{33} - |m_{23}|^2 = K_{z\zeta}K_{z\zeta} - |K_{z\zeta}|^2_{(z,\zeta)} = -|K_{z\zeta}(z, z)|^2.$$

Hence $K_{z\zeta}(z, z) = 0$, and so $w_z(z) = 0$. I.e., the Cauchy-Riemann Equations hold in $D$, and $w(z)$ is analytic in $D$.

In order to remove the assumption $w \in C^2$, we use a standard mollification argument. In a neighborhood of $z_0 \in D$, we approximate the continuous function $w(z)$ by mollified functions $w_\delta(z)$, such that $w_\delta \in C^3$ and $w_\delta \to w$ uniformly in a neighborhood of $z_0$. Since the property $P_3 \geq 0$ is additive and positive-homogeneous in $w$, we see also that $P_3 \geq 0$ for each $w_\delta$ as well as for $w$. We therefore know that $w_\delta$ is analytic in a neighborhood of $z_0$. By uniform convergence, so is $w$. Since $z_0$ was arbitrary, $w(z)$ is analytic throughout $D$.

From the above proof, it is clear that the hypotheses in statement (1) are considerably stronger than they need be. First, the fact that only $m_{22}m_{33} - |m_{23}|^2 \geq 0$ was used means that $P_3 = (k_{ij})$ need only be nonnegative definite on the subspace $L_3 = \{(X_i) \in C^3 : \sum X_i = 0\}$ of complex dimension 2. For, in the notation of the proof of the lemma, the latter condition is equivalent to

$$\begin{pmatrix} \bar{k}_{11} & \bar{k}_{12} \\ \bar{k}_{21} & \bar{k}_{22} \end{pmatrix} \geq 0.$$

The analogous form of the lemma, in which $(K(u^i, u^j)) \geq 0$ on $L_{n+1}$ for $u^0, u^1, \ldots, u^n \in D \Rightarrow (K_{u^i v_j}(u, u)) \geq 0$, is similarly proved. Secondly, there is now no need for the values of $w(z)$ to lie in UHP. These two alterations mean that the analyticity result holds when $w(z)$ is a continuous "infinitesimal transformation" of the class of maps of
D satisfying $P_3 \geq 0$, i.e., $w(z) = \frac{\partial f_t(z)}{\partial t} \bigg|_{t=t_0}$, where $f_t, 0 \leq t \leq t_0$, is a family of functions in $D$ satisfying $P_3 \geq 0$ in $D$ for all $t$, and $f_0(z) = z$. The class of such $w(z)$ is in fact characterized by the condition $P_3 \geq 0$ on $L_3$ (and likewise for general $n$). The positivity hypothesis could also be weakened from a global condition to a local one, but since $D$ is arbitrary and analyticity is a local property, this would be a trivial alteration. To summarize, we state:

**Theorem 1.** Let $w(z)$ be a continuous function in an open subset $D$ of UHP. If, for all $z_1, z_2, z_3 \in D$, the $3 \times 3$ matrix of difference quotients $d_{ij} = \frac{w(z_i) - w(z_j)}{(z_i - z_j)}$ satisfies $(d_{ij}) \geq 0$ on the subspace $\{ (X_i) \in \mathbb{C}^3 : \sum X_i = 0 \}$, then $w(z)$ is analytic in $D$.

It should be noted here that result (1), in the weaker form, can also be easily proven from Pick's Theorem (below). However, the latter requires a proof that considerably more involved than that given here for Theorem 1.

The statement (2) gives a characterization of the class $P$ of "positive" functions, analytic in UHP with values in UHP. It says that all of Pick's conditions together imply that $w$ is the restriction to $D$ of a $P$ function. The proof depends on the following:

**Pick's Theorem.** If $z_1, \ldots, z_n, w_1, \ldots, w_n \in \text{UHP}$ and $P_n = (d_{ij}) \geq 0$ for the $n \times n$ matrix of difference quotients $d_{ij} = (w_i - \overline{w_j})/(z_i - \overline{z_j})$, then there is a function $f \in P$ for which $f(z_i) = w_i$ for $1 \leq i \leq n$.

Now if $w(z)$ is continuous in $D$ and $K(z, \zeta) = \{w(z) - w(\zeta)\}/(z - \zeta)$ is nonnegative definite (of infinite order) in $D$, we can choose a dense sequence $(z_i)$ from $D$ and apply Pick's Theorem for each $n$. Because $P$ is a normal family, the $P$ functions so gotten have a normally convergent subsequence, and the analytic limit agrees with $w$ in $D$. We thus obtain

**Theorem 2.** Let $w(z)$ be a continuous function in a domain $D \subset \text{UHP}$ with values in UHP. If $\{w(z) - w(\zeta)\}/(z - \zeta)$ is a nonnegative definite kernel in $D$, then $w$ is analytic in $D$ and has an analytic extension to UHP whose values are in UHP.

I wish to take this opportunity to express my deep gratitude for Prof. Loewner's guidance and my sorrow at his loss.

I wish to take this opportunity to express my sorrow at the loss of Professor Charles Loewner, who, as my thesis advisor, inspired the work represented in this paper.
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