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**AN EXTENDED FORM OF THE MEAN-ERGODIC THEOREM**

LYNN CLIFFORD KURTZ AND DON HARRELL TUCKER

## AN EXTENDED FORM OF THE MEAN-ERGODIC THEOREM

LYNN C. KURTZ AND DON H. TUCKER

Suppose  $X$  is a reflexive Banach space and  $V$  is a continuous linear operator in  $X$  such that  $\|V^n\| \leq M$  for some  $M(n=0, 1, 2, \dots)$ . If  $N$  is the null space of  $I - V$  and  $R$  is the closure of the range of  $I - V$ , then the mean-ergodic theorem states that

$$\lim_{n \rightarrow \infty} \frac{(I + V + \dots + V^{n-1})x}{n} = Px,$$

where  $P$  is the projection associated with  $N$  and  $R$ ; the convergence is in the norm of  $X$ . This is pointwise  $C_1$ -summability of the sequence  $\{V^k\}_{k=0}^{\infty}$  to  $P$ , and it suggests a similar theorem for more general Hausdorff summability methods. The purpose of this note is to demonstrate a wide class of operator-valued Hausdorff summability methods which contain the sequence  $\{V^k\}_{k=0}^{\infty}$  in their wirkfelder and sum it to certain transforms of the projection operator  $P$ . This result shows much more clearly the sense in which convergence actually has meaning for such a sequence  $\{V^k\}_{k=0}^{\infty}$ .

Denote by  $C(X)$  the space of  $X$ -valued continuous functions on  $[0, 1]$  and by  $T_1$  the bounded linear transformation from  $C(X)$  into  $X$  given by  $T_1(f) = \int_0^1 f(t) dt$ . The mean-ergodic theorem states that

$$T_1\left(\sum_{k=0}^n \binom{n}{k} (t^k (1-t)^{n-k} V^k \cdot x)\right) \xrightarrow{n \rightarrow \infty} T_1(P \cdot x).$$

In this setting, the main theorem of this paper states a much stronger type of convergence; namely, that for any bounded linear operator  $T$  from  $C(X)$  into a Banach space  $Y$  such that the generating function for  $T$  is continuous at 0 and 1, it is true that

$$T\left(\sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k \cdot x\right) \xrightarrow{n \rightarrow \infty} T(P \cdot x).$$

In general one cannot expect much in the way of further relaxations on the operators  $T$ , i.e., on the functions which generate such operators. For example if the condition of continuity at 1 is removed, then this allows a generating function  $K(t) = 0$  for  $t < 1$ ,  $K(1) = 1$  and this generates the Hausdorff method corresponding to ordinary convergence. In general the sequence  $\{V^k \cdot x\}$  does not converge.

A nice presentation of the mean ergodic theorem as stated above is to be found in Lorch [2, pp. 54-56]. Suppose  $Y$  is a Banach space

and  $\mu = \{\mu_k\}_{k=0}^\infty$  is a sequence of elements of  $B[X, Y]$  such that the Hausdorff method  $H = \rho\mu\rho$  generated by  $\mu$  is regular relative to some  $L \in B[X, Y]$ . (See [1] for notation and terminology. Reference 8 in [1] is reference [3] of this paper.) It follows from [1] that there exists a function  $K$  on  $[0, 1]$  with values in  $B^+[X, Y]$  such that  $K$  satisfies the Gowurin  $\omega$ -property,

$$K(0) = 0, K(1) = L \quad \text{and} \quad \mu_n = \int_0^1 dK(t) \cdot t^n \quad \text{for} \quad n = 0, 1, 2, \dots .$$

**THEOREM.** *If  $K$  is continuous at  $t = 0$  and  $t = 1$ , then  $\{V^k\}_{k=0}^\infty$  is pointwise  $H$ -summable to  $LP$ , i.e.,  $H_n\{V^k\} \cdot x$  converges in the norm of  $Y$  to  $LPx$  for each  $x \in X$ .*

The essential ingredient of the proof of the theorem is the following lemma.

**LEMMA.** *If  $\{s_k\}_{k=0}^\infty$  is a bounded sequence of elements of a linear normed space  $S$  and  $0 < a \leq t \leq b < 1$ , then*

$$\left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (s_k - s_{k+1}) \right\|_S$$

*converges uniformly to zero for  $t \in [a, b]$ .*

*Proof of the lemma.*<sup>1</sup> Suppose  $\|s_k\| \leq N'$  for  $k = 0, 1, 2, \dots$ , then set

$$\begin{aligned} A_n(t) &= \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (s_k - s_{k+1}) \\ &= \sum_{k=1}^n \left[ \binom{n}{k} t^k (1-t)^{n-k} - \binom{n}{k-1} t^{k-1} (1-t)^{n-k+1} \right] s_k \\ &\quad + (1-t)^n s_0 - t^n s_{n+1} \\ &= \sum_{k=1}^n \binom{n}{k} t^k (1-t)^{n-k} \left[ 1 - \frac{k}{n-k+1} \cdot \frac{1-t}{t} \right] s_k \\ &\quad + (1-t)^n s_0 - t^n s_{n+1} \\ &= \frac{1}{t} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left[ \frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k - t^n s_{n+1} . \end{aligned}$$

$$\|A_n(t)\|_S \leq \frac{1}{t} \left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left[ \frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k \right\|_S + t^n \|s_{n+1}\|$$

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<sup>1)</sup> The proof presented here is incorrect. See part 2 for a corrected proof.

where  $0 < a \leq t \leq b < 1$ , and hence

$$\|A_n(t)\|_s \leq \frac{N'}{t} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n} \cdot \frac{n}{n+1} \right|}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} + t^n \cdot N'.$$

Let  $f_n(x, t)$  be given by

$$f_n(x, t) = \left| x - t \cdot \frac{n}{n+1} \right| / \left( 1 - t \cdot \frac{n}{n+1} \right)$$

and  $C_n(t)$  by

$$C_n(t) = \frac{1}{t} B_n[f_n(x, t)]|_{x=t}$$

where  $B_n$  denotes the  $n$ -th Bernstein polynomial. The above inequality may now be written

$$\|A_n(t)\|_s \leq N' |C_n(t)| + t^n \cdot N'$$

and the second term converges uniformly to zero for  $t \in [a, b]$ .

The first term is treated as follows. By a direct calculation it can be shown that for each  $x \in [0, b]$ , the collection  $\{f_n(x, t)\}$  is equi-uniformly continuous in  $t$  for  $t \in [0, b]$ , that is to say, if  $\varepsilon > 0$ , then there exists  $\delta > 0$  such that  $|f_n(x, s) - f_n(x, t)| < \varepsilon/2$  for all  $s, t \in [0, b]$  such that  $|s - t| < \delta$  and for all  $n$ .

Consider a fixed  $t \in [0, b]$  and set  $A = \{k: |k/n - t| < \delta\}$  and  $B = \{0, 1, \dots, n\} - A$ . Then

$$\begin{aligned} & |B_n[f_n(x, t)] - f_n(x, t)| \\ & \leq \left( \sum_A + \sum_B \right) \left| \binom{n}{k} t^k (1-t)^{n-k} \left\{ \frac{x - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right\} - f_n(x, t) \right| \\ & \sum_A \leq \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

Set  $Q = \max_{0 \leq t, x \leq b} f_n(x, t)$  for  $n = 0, 1, 2, \dots$  and the second term can be treated as follows:

$$\sum_B \leq 2Q \sum_B \binom{n}{k} t^k (1-t)^{n-k} \frac{(k - nt)^2}{n^2 \delta^2} \leq \frac{2Q}{n^2 \delta^2} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (k - nt)^2$$

which, as is well known, converges uniformly to zero for  $t \in [0, 1]$ . Hence, there exists an integer  $N_0$  such that  $\sum_B < \varepsilon/2$  for  $n > N_0$  and further such that  $|B_n[f_n(x, t)] - f_n(x, t)| < \varepsilon$  for  $n > N_0$ , both

inequalities holding uniformly for  $0 \leqq t \leqq b$ . Collecting all these items together yields

$$\lim_{n \rightarrow \infty} C_u(t) = \frac{1}{t} \frac{|t - t|}{1 - t} = 0$$

uniformly for  $t \in [a, b]$ , and hence  $\|A_n(t)\|_S \rightarrow 0$  uniformly on  $[a, b]$ .

*Proof of the theorem.* Let

$$T_n = H_n \{V^k\}_{k=0}^\infty = \sum_{k=0}^n \binom{n}{k} \Delta^{n-k} \mu_k V^k = \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k \right].$$

Since  $N, R$  is a complementary pair in  $X$ , it is sufficient to investigate the behavior of  $T_n$  on each of these sets.

Suppose  $f \in N$ , i.e.,  $Vf = f$ , then

$$\begin{aligned} T_n f &= \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k f \right] = \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} f \right] \\ &= \int_0^1 dK(t) \cdot f = [K(1) - K(0)]f = Lf = LPf. \end{aligned}$$

Now suppose  $f \in R$  and  $\varepsilon > 0$ , then there exists  $g$  and  $h$  such that  $f = g - Vg + h$  where  $\|h\| < \varepsilon/4[1 + W_0^1 K]M$ . For this  $f$ ,

$$\begin{aligned} T_n f &= \int_0^1 dK(t) \cdot \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \\ &\quad + \int_0^1 dK(t) \cdot \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} V^k h = I + II. \\ \|II\|_Y &\leqq W_0^1 K \cdot \max_{0 \leqq t \leqq 1} \left| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \right| \max_{0 \leqq k \leqq n} \|V^k h\|_X \\ &\leqq W_0^1 K \cdot M \cdot \varepsilon/4 [1 + W_0^1 K]M < \frac{\varepsilon}{4} \quad \text{for all } n. \end{aligned}$$

$$\begin{aligned} \|I\|_Y &= \|I\|_{Y^{**}} = \left\| \int_0^a + \int_a^b + \int_b^1 \right\|_{Y^{**}} \\ &\leqq \left\| \int_0^a \right\|_{Y^{**}} + \left\| \int_a^b \right\|_{Y^{**}} + \left\| \int_b^1 \right\|_{Y^{**}}. \end{aligned}$$

It is necessary to regard the norms on the right as  $Y^{**}$  norms because these integrals may exist only as elements in  $Y^{**}$  and not as elements in  $Y$  (see the remarks following Theorem 1 [3, p. 950].)

$$\left\| \int_0^a dK(t) \cdot \left[ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leqq W_0^a K \cdot 2M \cdot \|g\|_X$$

and

$$\left\| \int_b^1 dK(t) \cdot \left[ \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leqq W_b^1 K \cdot 2M \cdot \|g\|_X.$$

Since  $K$  is assumed continuous at  $t = 0$  and  $t = 1$ , there are values for  $a$  and  $b$  sufficiently near, but distinct from 0 and 1 respectively, such that each of  $W_0^a K$  and  $W_1^b K$  less than  $\varepsilon/8M[1 + \|g\|]$ . With these values of  $a$  and  $b$ , there is  $n$  sufficiently large, by the above lemma, that

$$\max_{a \leq t \leq b} \left\| \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} [V^k g - V^{k+1} g] \right\|_X \leq \varepsilon/2[1 + W_a^b K].$$

Collecting all this together yields

$$\|T_n f\|_Y \leq \varepsilon$$

for all  $n$  sufficiently large. Thus

$$\lim_{n \rightarrow \infty} \|T_n f\|_Y = \lim_{n \rightarrow \infty} \|T_n f - LPf\| = 0$$

since

$$LPf = \theta_Y$$

and this completes the proof.

In case that  $Y \equiv X$  and  $H$  is regular relative to  $I$ , then  $H$  sums  $\{V^k\}_{k=0}^\infty$  to  $P$ . In particular, any regular scalar-valued Hausdorff method whose generating function  $K$  is continuous at  $t = 0$  and  $t = 1$  will sum  $\{V^k\}_{k=0}^\infty$  to  $P$ . The case treated in [2], corresponds to the case here where  $K(t) = tI$ , i.e., the  $C_1$  method. The following example illustrates the theorem for a nonscalar-valued Hausdorff method.

Suppose  $X = Y = H$ , a Hilbert space. Suppose also that  $K$  is a bounded resolution of the identity such that  $K(0) = 0, K(1) = I, K$  is continuous at 0 and 1 in the operator norm, and  $K$  satisfies the Gowurin  $\omega$ -property. The approximating sums for integrals of the form  $\int_0^1 t^n dK(t)$  converge to the integral in the operator norm [2], hence they converge in the sense given by Tucker [3]. Consider the moment sequence  $\{\mu_n\}_{n=0}^\infty$  given by  $\mu_n = \int_0^1 t^n dK(t)$ . As shown in [2],  $\mu_1$  is a self-adjoint operator in  $H$ , and if we denote it by  $A$ , it follows that  $\mu_n = A^n (n = 0, 1, 2, \dots)$  where  $\mu_0 = K(1) = A^0 = I$ . If  $\{V^n\}_{n=0}^\infty$  is a sequence of operators as given in the theorem, and  $H(\mu)$  is the Hausdorff summability method generated by  $\{\mu_n\} = \{A^n\}$ , then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k} (A^{n-k} A^k) V^k x = Px$$

for all  $x \in H$ , the limit being taken in the norm of  $H$ .

## PART 2

It has been pointed out that the proof of the lemma given above is incorrect. It can be corrected in the following manner. As given,

$$\|A_n(t)\|_s \leq \frac{N'}{t} \sum_{k=0}^n \left( \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) + t^n N'.$$

Proceed as follows. For  $0 < a \leq t \leq b < 1$

$$\|A_n(t)\|_s \leq \frac{N'}{a} \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} + b^n N'.$$

Suppose  $\varepsilon > 0$  and pick  $\delta$  such that  $0 < \delta < \{(1-b)\varepsilon/2/(1+\varepsilon/2)\}$ . For  $t \in [a, b]$ , set

$$A_t = \left\{ k: \left| t - \frac{k}{n+1} \right| < \delta \right\} \quad \text{and} \quad B_t = \left\{ k: \left| t - \frac{k}{n+1} \right| \geq \delta \right\}.$$

Then

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \\ &= \left( \sum_{A_t} + \sum_{B_t} \right) \left\{ \binom{n}{k} t^k (1-t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right\}. \end{aligned}$$

Consider the sums separately.

$$\begin{aligned} \sum_{A_t} &\leq \sum_{A_t} \binom{n}{k} t^k (1-t)^{n-k} \frac{\delta}{1-b-\delta} \leq \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \\ \sum_{B_t} &= \frac{1}{1-t} \sum_{B_t} \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1} \left| t - \frac{k}{n+1} \right|. \end{aligned}$$

For  $k \in B_t$ ,

$$\left| \frac{k}{n+1} - t \right| \leq 1 \leq \frac{((n+1)t - k)^2}{\delta^2(n+1)^2},$$

so

$$\begin{aligned}
\sum_{b_t} &\leq \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^n \binom{n}{k} \frac{n+1}{n+1-k} t^k (1-t)^{n-k+1} [(n+1)t-k]^2 \\
&= \frac{1}{(1-t)\delta^2(n+1)^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1} [(n+1)t-k]^2 \\
&= \frac{1}{(1-t)\delta^2} \sum_{k=0}^{n+1} \binom{n+1}{k} t^k (1-t)^{n-k+1} \left(t - \frac{k}{n+1}\right)^2 \\
&= \frac{1}{(1-t)\delta^2} \cdot \frac{t(1-t)}{n+1} \leq \frac{b}{(n+1)\delta^2}.
\end{aligned}$$

Collecting this together gives

$$\|A_n(t)\|_s \leq \frac{N'}{a} \left( \frac{\varepsilon}{2} + \frac{b}{(n+1)\delta^2} + b^n N' \right), \quad \text{for } 0 < a \leq t \leq b < 1,$$

which proves the lemma.

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