AN EXTENDED FORM OF THE MEAN-ERGODIC THEOREM

LYNN CLIFFORD KURTZ AND DON HARRELL TUCKER
Suppose $X$ is a reflexive Banach space and $V$ is a continuous linear operator in $X$ such that $||V^n|| \leq M$ for some $M(n=0, 1, 2, ...)$ If $N$ is the null space of $I-V$ and $R$ is the closure of the range of $I-V$, then the mean-ergodic theorem states that

$$\lim_{n\to\infty} \frac{(I+V+\cdots+V^{n-1})x}{n} = Px,$$

where $P$ is the projection associated with $N$ and $R$; the convergence is in the norm of $X$. This is pointwise $C_1$-summability of the sequence $\{V^n\}_{k=0}^\infty$ to $P$, and it suggests a similar theorem for more general Hausdorff summability methods. The purpose of this note is to demonstrate a wide class of operator-valued Hausdorff summability methods which contain the sequence $\{V^n\}_{k=0}^\infty$ in their workfelder and sum it to certain transforms of the projection operator $P$. This result shows much more clearly the sense in which convergence actually has meaning for such a sequence $\{V^n\}_{k=0}^\infty$.

Denote by $C(X)$ the space of $X$-valued continuous functions on $[0,1]$ and by $T_1$ the bounded linear transformation from $C(X)$ into $X$ given by $T_1(f) = \int_a^bf(t)dt$. The mean-ergodic theorem states that

$$T_1\left(\sum_{k=0}^n \binom{n}{k} t^k(1-t)^{n-k} V^k \cdot x\right) \xrightarrow[n\to\infty]{\text{a.e.}} T_1(P\cdot x).$$

In this setting, the main theorem of this paper states a much stronger type of convergence; namely, that for any bounded linear operator $T$ from $C(X)$ into a Banach space $Y$ such that the generating function for $T$ is continuous at 0 and 1, it is true that

$$T\left(\sum_{k=0}^n \binom{n}{k} t^k(1-t)^{n-k} V^k \cdot x\right) \xrightarrow[n\to\infty]{\text{a.e.}} T(P\cdot x).$$

In general one cannot expect much in the way of further relaxations on the operators $T$, i.e., on the functions which generate such operators. For example if the condition of continuity at 1 is removed, then this allows a generating function $K(t) = 0$ for $t < 1$, $K(1) = 1$ and this generates the Hausdorff method corresponding to ordinary convergence. In general the sequence $\{V^k \cdot x\}$ does not converge.

A nice presentation of the mean ergodic theorem as stated above is to be found in Lorch [2, pp. 54–56]. Suppose $Y$ is a Banach space...
and $\mu = \{\mu_k\}_{k=0}^\infty$ is a sequence of elements of $B[X, Y]$ such that the Hausdorff method $H = \rho \mu \rho$ generated by $\mu$ is regular relative to some $L \in B[X, Y]$. (See [1] for notation and terminology. Reference 8 in [1] is reference [3] of this paper.) It follows from [1] that there exists a function $K$ on $[0,1]$ with values in $B^+[X, Y]$ such that $K$ satisfies the Gowurin $\omega$-property,

$$K(0) = 0, K(1) = L \quad \text{and} \quad \mu_n = \int_0^1 dK(t) \cdot t^n \quad \text{for} \quad n = 0, 1, 2, \cdots.$$ 

**THEOREM.** If $K$ is continuous at $t = 0$ and $t = 1$, then $\{V_k\}_{k=0}^\infty$ is pointwise $H$-summable to $L^p$, i.e., $H_n\{V_k\} \cdot x$ converges in the norm of $Y$ to $L^p_x$ for each $x \in X$.

The essential ingredient of the proof of the theorem is the following lemma.

**LEMMA.** If $\{s_k\}_{k=0}^\infty$ is a bounded sequence of elements of a linear normed space $S$ and $0 < a \leq t \leq b < 1$, then

$$\left\| \sum_{k=0}^n \left( \binom{n}{k} t^k (1 - t)^{n-k} (s_k - s_{k+1}) \right) \right\|_S$$

converges uniformly to zero for $t \in [a, b]$.

**Proof of the lemma.** Suppose $\| s_k \| \leq N'$ for $k = 0, 1, 2, \cdots$, then set

$$A_n(t) = \sum_{k=0}^n \left( \binom{n}{k} t^k (1 - t)^{n-k} (s_k - s_{k+1}) \right)$$

$$= \sum_{k=1}^n \left[ \left( \binom{n}{k} t^k (1 - t)^{n-k} \right) \left( \frac{n}{k-1} \right) t^{k-1} (1 - t)^{n-k+1} \right] s_k$$

$$+ (1 - t)^n s_0 - t^n s_{n+1}$$

$$= \sum_{k=1}^n \left( \binom{n}{k} t^k (1 - t)^{n-k} \left[ 1 - \frac{k}{n-k+1} \cdot \frac{1-t}{t} \right] s_k \right)$$

$$+ (1 - t)^n s_0 - t^n s_{n+1}$$

$$= \frac{1}{t} \sum_{k=0}^n \left( \binom{n}{k} t^k (1 - t)^{n-k} \left[ \frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k - t^n s_{n+1} \right).$$

$$\| A_n(t) \|_S \leq \frac{1}{t} \left\| \sum_{k=0}^n \left( \binom{n}{k} t^k (1 - t)^{n-k} \left[ \frac{t - \frac{k}{n} \cdot \frac{n}{n+1}}{1 - \frac{k}{n} \cdot \frac{n}{n+1}} \right] s_k \right) + t^n \| s_n+1 \| \right\|_S$$

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1) The proof presented here is incorrect. See part 2 for a corrected proof.
where \(0 < a \leq t \leq b < 1\), and hence

\[
\| A_n(t) \|_S \leq \frac{N'}{t} \sum_{k=0}^n \left( \frac{n}{k} \right) t^k (1 - t)^n - k \left| \frac{t - \frac{k}{n} \frac{n}{n + 1}}{1 - \frac{k}{n} \frac{n}{n + 1}} \right| + t^n \cdot N'.
\]

Let \(f_n(x, t)\) be given by

\[
f_n(x, t) = \left| x - \frac{n}{n + 1} \right| \left( 1 - \frac{n}{n + 1} \right)
\]

and \(C_n(t)\) by

\[
C_n(t) = \frac{1}{t} B_n[f_n(x, t)]_{x=t}
\]

where \(B_n\) denotes the \(n\)-th Bernstein polynomial. The above inequality may now be written

\[
\| A_n(t) \|_S \leq N' \cdot \| C_n(t) \| + t^n \cdot N'
\]

and the second term converges uniformly to zero for \(t \in [a, b]\).

The first term is treated as follows. By a direct calculation it can be shown that for each \(x \in [0, b]\), the collection \(\{f_n(x, t)\}\) is equi-uniformly continuous in \(t\) for \(t \in [0, b]\), that is to say, if \(\varepsilon > 0\), then there exists \(\delta > 0\) such that \(|f_n(x, s) - f_n(x, t)| < \varepsilon/2\) for all \(s, t \in [0, b]\) such that \(|s - t| < \delta\) and for all \(n\).

Consider a fixed \(t \in [0, b]\) and set \(A = \{k: |k/n - t| < \delta\}\) and \(B = \{0, 1, \ldots, n\} - A\). Then

\[
|B_n[f_n(x, t)] - f_n(x, t)| \\
\leq (\sum_{\delta} + \sum_{\delta}) \left| \frac{n}{k} t^k (1 - t)^n - k \left( x - \frac{k}{n} \frac{n}{n + 1} \right) \right|
\]

\[
\sum_{\delta} \leq \sum_{k=0}^n \left( \frac{n}{k} \right) t^k (1 - t)^n - k \cdot \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.
\]

Set \(Q = \max_{x \in [a, b]} |f_n(x, t)|\) for \(n = 0, 1, 2, \ldots\) and the second term can be treated as follows:

\[
\sum_{\delta} \leq 2Q \sum_{\delta} \left( \frac{n}{k} \right) t^k (1 - t)^n - k \frac{(k - nt)^2}{n^2 \delta^2} \leq \frac{2Q}{n^2 \delta^2} \sum_{k=0}^n \left( \frac{n}{k} \right) t^k (1 - t)^n - k \frac{(k - nt)^2}{n^2 \delta^2}
\]

which, as is well known, converges uniformly to zero for \(t \in [0, 1]\). Hence, there exists an integer \(N_0\) such that \(\sum_{\delta} < \varepsilon/2\) for \(n > N_0\) and further such that \(|B_n[f_n(x, t)] - f_n(x, t)| < \varepsilon\) for \(n > N_0\), both
inequalities holding uniformly for $0 \leq t \leq b$. Collecting all these items together yields

$$\lim_{n \to \infty} C_n(t) = \frac{1}{t} \frac{|t - t|}{1 - t} = 0$$

uniformly for $t \in [a, b]$, and hence $\|A_n(t)\|_s \to 0$ uniformly on $[a, b]$.

**Proof of the theorem.** Let

$$T_n = H_n\{V^k\}_k^\infty = \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^{n-k} \mu_k V^k = \int_0^1 dK(t) \left[ \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} V^k \right] .$$

Since $N, R$ is a complementary pair in $X$, it is sufficient to investigate the behavior of $T_n$ on each of these sets.

Suppose $f \in N$, i.e., $Vf = f$, then

$$T_n f = \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} V^k f \right] = \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} f \right] = \int_0^1 dK(t) \cdot f = [K(1) - K(0)] f = Lf = LF f .$$

Now suppose $f \in R$ and $\varepsilon > 0$, then there exists $g$ and $h$ such that $f = g - Vg + h$ where $\|h\| < \varepsilon/4[1 + W_0^1 K]M$. For this $f$,

$$T_n f = \int_0^1 dK(t) \cdot \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} [V^k g - V^{k+1} g] + \int_0^1 dK(t) \cdot \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} V^k h = I + II .$$

$$\|II\|_Y \leq W_0^1 K \cdot \max_{0 \leq k \leq n} \left| \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} \right| \max_{0 \leq k \leq n} \|V^k h\|_x \leq W_0^1 K \cdot M \cdot \varepsilon/4[1 + W_0^1 K]M < \frac{\varepsilon}{4} \text{ for all } n .$$

$$\|I\|_Y = \|I\|_{Y^{**}} = \left\| \int_0^a + \int_a^b \right\|_{Y^{**}} \leq \left\| \int_0^a \right\|_{Y^{**}} + \left\| \int_a^b \right\|_{Y^{**}} .$$

It is necessary to regard the norms on the right as $Y^{**}$ norms because these integrals may exist only as elements in $Y^{**}$ and not as elements in $Y$ (see the remarks following Theorem 1 [3, p. 950].)

$$\left\| \int_0^a dK(t) \cdot \left[ \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leq W_0^a K \cdot 2M \cdot \|g\|_x$$

and

$$\left\| \int_0^1 dK(t) \cdot \left[ \sum_{k=0}^\infty \left( \begin{array}{c} n \\ k \end{array} \right) t^k(1-t)^{n-k} [V^k g - V^{k+1} g] \right] \right\|_{Y^{**}} \leq W_0^b K \cdot 2M \cdot \|g\|_x .$$
Since $K$ is assumed continuous at $t = 0$ and $t = 1$, there are values for $a$ and $b$ sufficiently near, but distinct from $0$ and $1$ respectively, such that each of $W_o^aK$ and $W_b^bK$ less than $\varepsilon/8M[1 + \|g\|]$. With these values of $a$ and $b$, there is $n$ sufficiently large, by the above lemma, that

$$\max_{a \leq t \leq b} \left\| \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} [V^k g - V^{k+1} g] \right\|_x \leq \varepsilon/2[1 + W^a_oK].$$

Collecting all this together yields

$$\| T_n f \|_Y \leq \varepsilon$$

for all $n$ sufficiently large. Thus

$$\lim_{n \to \infty} \| T_n f \|_Y = \lim_{n \to \infty} \| T_n f - LPf \| = 0$$

since

$$LPf = \theta_Y$$

and this completes the proof.

In case that $Y = X$ and $H$ is regular relative to $I$, then $H$ sums $\{V^k\}_{k=0}^\infty$ to $P$. In particular, any regular scalar-valued Hausdorff method whose generating function $K$ is continuous at $t = 0$ and $t = 1$ will sum $\{V^k\}_{k=0}^\infty$ to $P$. The case treated in [2], corresponds to the case here where $K(t) = tI$, i.e., the $C_1$ method. The following example illustrates the theorem for a nonscalar-valued Hausdorff method.

Suppose $X = Y = H$, a Hilbert space. Suppose also that $K$ is a bounded resolution of the identity such that $K(0) = 0, K(1) = I, K$ is continuous at $0$ and $1$ in the operator norm, and $K$ satisfies the Gowerin $\omega$-property. The approximating sums for integrals of the form $\int_0^1 t^n dK(t)$ converge to the integral in the operator norm [2], hence they converge in the sense given by Tucker [3]. Consider the moment sequence $\{\mu_n\}_{n=0}^\infty$ given by $\mu_n = \int_0^1 t^n dK(t)$. As shown in [2], $\mu_n$ is a self-adjoint operator in $H$, and if we denote it by $A_n$, it follows that $\mu_n = A^n (n = 0, 1, 2, \cdots)$ where $\mu_0 = K(1) = A^0 = I$. If $\{V^x\}_{n=0}^\infty$ is a sequence of operators as given in the theorem, and $H(\mu)$ is the Hausdorff summability method generated by $\{\mu_n\} = \{A^n\}$, then

$$\lim \sum_{n=0}^{\infty} \binom{n}{k} (A^{n-k} A^k) V^k x = Px$$

for all $x \in H$, the limit being taken in the norm of $H$. 
PART 2

It has been pointed out that the proof of the lemma given above is incorrect. It can be corrected in the following manner. As given,

\[ \| A_n(t) \|_s \leq \frac{N'}{t} \sum_{k=0}^{n} \left( \binom{n}{k} t^k (1 - t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) + t^n N'. \]

Proceed as follows. For \( 0 < a \leq t \leq b < 1 \)

\[ \| A_n(t) \|_s \leq \frac{N'}{a} \sum_{k=0}^{n} \left( \binom{n}{k} t^k (1 - t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) + b^n N'. \]

Suppose \( \varepsilon > 0 \) and pick \( \delta \) such that \( 0 < \delta < \{ (1 - b)\varepsilon/2 / (1 + \varepsilon/2) \} \).

For \( t \in [a, b] \), set

\[ A_t = \{ k : \left| t - \frac{k}{n+1} \right| < \delta \} \quad \text{and} \quad B_t = \{ k : \left| t - \frac{k}{n+1} \right| \geq \delta \}. \]

Then

\[ \sum_{k=0}^{n} \left( \binom{n}{k} t^k (1 - t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right) \]

\[ = \left( \sum_{A_t} + \sum_{B_t} \right) \left\{ \binom{n}{k} t^k (1 - t)^{n-k} \frac{\left| t - \frac{k}{n+1} \right|}{1 - \frac{k}{n+1}} \right\}. \]

Consider the sums separately.

\[ \sum_{A_t} \leq \sum_{A_t} \left( \binom{n}{k} t^k (1 - t)^{n-k} \frac{\delta}{1 - b - \delta} \right) \leq \sum_{k=0}^{n} \left( \binom{n}{k} t^k (1 - t)^{n-k} \frac{\varepsilon}{2} \right) = \frac{\varepsilon}{2}. \]

\[ \sum_{B_t} = \frac{1}{1 - t} \sum_{B_t} \left( \binom{n}{k} \frac{n + 1}{n + 1 - k} t^k (1 - t)^{n-k+1} \left| t - \frac{k}{n+1} \right| \right). \]

For \( k \in B_t \),

\[ \left| \frac{k}{n+1} - t \right| \leq 1 \leq \frac{(n + 1)t - k^2}{\delta^2 (n + 1)^2}, \]
\[
\begin{align*}
\sum_{n_t} & \leq \frac{1}{(1 - t)\delta^2(n + 1)^2} \sum_{k=0}^{n} \binom{n}{k} \frac{n + 1}{n + 1 - k} t^k(1 - t)^{n-k+1}[(n + 1)t - k]^2 \\
& = \frac{1}{(1 - t)\delta^2(n + 1)^2} \sum_{k=0}^{n+1} \binom{n + 1}{k} t^k(1 - t)^{n-k+1}[(n + 1)t - k]^2 \\
& = \frac{1}{(1 - t)\delta^2} \sum_{k=0}^{n+1} \binom{n + 1}{k} t^k(1 - t)^{n-k+1} \left( t - \frac{k}{n + 1} \right)^2 \\
& = \frac{1}{(1 - t)\delta^2} \cdot \frac{t(1 - t)}{n + 1} \leq \frac{b}{(n + 1)\delta^2}. 
\end{align*}
\]

Collecting this together gives

\[
\| A_n(t) \|_s \leq \frac{N'}{a} \left( \frac{\varepsilon}{2} + \frac{b}{(n + 1)\delta^2} + b^n N' \right), \quad \text{for } 0 < a \leq t \leq b < 1,
\]

which proves the lemma.

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Charles A. Akemann, *Invariant subspaces of C(G)* .......... 421
Dan Amir and Zvi Ziegler, *Generalized convexity cones and their duals* .... 425
Raymond Balbes, *On (J, M, m)-extensions of order sums of distributive lattices* ......................................................... 441
Jan-Erik Björk, *Extensions of the maximal ideal space of a function algebra* ................................................................. 453
Frank Castagna, *Sums of automorphisms of a primary abelian group* .... 463
Theodore Seio Chihara, *On determinate Hamburger moment problems* .... 475
Zeev Ditzian, *Convolution transforms whose inversion function has complex roots in a wide angle* ....................................... 485
Myron Goldstein, *On a paper of Rao* ........................................ 497
Velmer B. Headley and Charles Andrew Swanson, *Oscillation criteria for elliptic equations* .................................................. 501
John Willard Heidel, *Qualitative behavior of solutions of a third order nonlinear differential equation* ................................. 507
Alan Carleton Hindmarsh, *Pick’s conditions and analyticity* ............ 527
Bruce Ansgar Jensen and Donald Wright Miller, *Commutative semigroups which are almost finite* ........................................... 533
Lynn Clifford Kurtz and Don Harrell Tucker, *An extended form of the mean-ergodic theorem* .................................................. 539
S. P. Lloyd, *Feller boundary induced by a transition operator* ............ 547
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, *A new proof of the maximum principle for doubly-harmonic functions* ........ 567
Robert Einsohn Mosher, *The product formula for the third obstruction* .... 573
Sam Bernard Nadler, Jr., *Sequences of contractions and fixed points* .... 579
Eric Albert Nordgren, *Invariant subspaces of a direct sum of weighted shifts* ................................................................. 587
Fred Richman, *Thin abelian p-groups* ........................................ 599
Jordan Tobias Rosenbaum, *Simultaneous interpolation in H^2, II* ........ 607
Charles Thomas Scarborough, *Minimal Urysohn spaces* .................... 611
Malcolm Jay Sherman, *Disjoint invariant subspaces* ......................... 619
Joel John Westman, *Harmonic analysis on groupoids* ....................... 621
William Jennings Wickless, *Quasi-isomorphism and TFM rings* ......... 633
Minoru Hasegawa, *Correction to “On the convergence of resolvents of operators”* ............................................................. 641