

Pacific Journal of Mathematics

THE PRODUCT FORMULA FOR THE THIRD OBSTRUCTION

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Let ξ be an $SO(n)$ -bundle with $n > 3$; let $p: E \rightarrow B$ be the projection in the associated $(n - 1)$ -sphere bundle. In this note we express the third obstruction to a cross-section of p as a tertiary characteristic class and prove a product formula for the behavior of this class under Whitney sum.

The first obstruction is the Euler class $\chi(\xi) \in H^n(B; Z)$. χ is a primary characteristic class and satisfies $\chi = j^*(U)$, where $j: B \rightarrow T$ is the inclusion into the Thom space and $U \in H^n(T; Z)$ is the Thom class. Whenever $\chi(\xi) = 0$, a secondary characteristic class

$$\alpha(\xi) \in H^{n+1}(B; Z_2)/(Sq^2 + w_2 \smile)H^{n-1}(B; Z)$$

is defined. α is the second obstruction and satisfies

$$\alpha = (Sq^2 + w_2 \smile)_j(U).$$

Thus α is obtained by applying a twisted functional primary operation to U . The third obstruction $\gamma(\xi)$, defined whenever $\alpha(\xi) \equiv 0$, will be expressed as the value $\Phi_j(U)$ of a certain twisted functional secondary operation.

It is immediately plausible to consider as $(n + 1)$ -ary characteristic classes the values of certain functional twisted n -ary operations on U , defined when appropriate n -ary characteristic classes vanish. We hope to deal with such classes systematically in a future paper, but the treatment is expected to be more complicated technically; hence $\gamma(\xi)$ is presented here as an illustrative example in a straightforward setting.

The paper is organized as follows. Section 2 is a statement of results, while in § 3 we define $\gamma(\xi)$. The Peterson-Stein formula and the proof of (2.2) appears in § 4; the product formula is obtained in § 5. We conclude in § 6 with an example.

Throughout the paper all cohomology is taken with Z_2 as coefficients unless otherwise indicated.

2. Statement of results. Suppose ξ is an $SO(n)$ -bundle with $n > 3$ and suppose $\chi(\xi) = 0$. Let

$$\alpha(\xi) \in H^{n+1}(B)/(Sq^2 + w_2 \smile)H^{n-1}(B; Z)$$

be the secondary characteristic class given by $\alpha(\xi) = (Sq^2 + w_2 \smile)_j(U)$

[5, 6, 7, 9]. By [9], $\alpha(\xi)$ is the second obstruction to a cross-section in the associated sphere bundle.

Suppose now $\alpha(\xi) \equiv 0$. Then in §3 is defined a tertiary characteristic class $\gamma(\xi) \in H^{n+2}(B)$ modulo an indeterminacy Q , given in (3.6). γ is natural in the following sense.

PROPOSITION 2.1. $f: \xi' \rightarrow \xi$ be a map of $SO(n)$ -bundles. Suppose $\gamma(\xi)$ is defined. Then $\gamma(\xi') \equiv f^*(\gamma(\xi)) \bmod Q(\xi')$.

In §4 we establish the following.

PROPOSITION 2.2. $\gamma(\xi)$ is the third obstruction to a cross-section of p .

For product formulas we now assume ξ and ξ' are $SO(n)$ and $SO(n')$ -bundles over B and B' respectively such that $\alpha(\xi)$ and $\alpha(\xi')$ are defined. Let $\xi \oplus \xi'$ be the external Whitney sum over $B \times B'$. By the Whitney formula for secondary characteristic classes [9], $\alpha(\xi \oplus \xi') \equiv 0$ and thus $\gamma(\xi \oplus \xi')$ is defined. In §5 we prove the following.

PROPOSITION 2.3. $\gamma(\xi \oplus \xi') \equiv \alpha(\xi) \otimes \alpha(\xi')$ modulo the total indeterminacy.

Taking $B = B'$ and writing $\xi + \xi'$ for the internal Whitney sum, we obtain the following corollary to (2.1) and (2.3).

PROPOSITION 2.4. $\gamma(\xi + \xi') \equiv \alpha(\xi) \smile \alpha(\xi')$ modulo the total indeterminacy.

3. Definition of $\gamma(\xi)$. Let A be the mod 2 Steenrod algebra. In the semi-tensor product $H^*(BSO) \odot A$ [3] we have, in the terminology of [11], the relation

$$(3.1) \quad (1 \otimes Sq^2 + w_2 \otimes 1)(1 \otimes Sq^2 + w_2 \otimes 1) = 0$$

over Z . Let $\beta = 1 \otimes Sq^2 + w_2 \otimes 1$. According to [4] and [11], (3.1) defines for each n sufficiently large ($n > 2$ suffices in this case) a twisted secondary operation $\Phi^{(n)}$. $\Phi^{(n)}$ is defined on an n -dimensional integral cohomology class x of a space X , where $\beta x = 0$ and $H^*(BSO) \times A$ acts on the cohomology of X via a vector bundle. The indeterminacy of $\Phi^{(n)}(X)$ is the subgroup $\beta H^{n+1}(X)$ of $H^{n+3}(X)$. While $\Phi^{(n)}$ is not uniquely determined by (3.1), computation in the universal example verifies the following for $n > 2$.

PROPOSITION 3.2. For each n , there exist precisely two distinct

operations $\Phi_1^{(n)}$ and $\Phi_2^{(n)}$ associated with (3.1); these operations are related by $\Phi_1^{(n)}(x) + \Phi_2^{(n)}(x) = Sq^3x = w_3 \smile x$.

Let U_n be the Thom class of the universal $SO(n)$ -bundle γ_n . Another calculation checks the following.

PROPOSITION 3.3. For each n , there is a unique choice of $\Phi^{(n)}$ such that $\Phi^{(n)}(U_n) = 0$.

We now assume that $\Phi^{(n)}$ are so chosen and further note that $\Phi^{(n)}$ so chosen are compatible with coboundary, as is verified by consideration of the natural map $T(\gamma_{n-1} + 1) \rightarrow T(\gamma_n)$ of Thom spaces.

Suppose now the $SO(n)$ -bundle ξ satisfies $\chi(\xi) = 0$ and $\alpha(\xi) \equiv 0$. Then U satisfies $j^*(U) = 0$, $\beta(U) = 0$, $\beta_j(U) \equiv 0$, and $\Phi(U) = 0$ with zero indeterminacy. Under these circumstances one defines $\Phi_j(U)$ by the analogue for twisted operations of Peterson's generalization [8] of Steenrod's basic method [10], detailed below; one then defines $\gamma(\xi)$ as follows.

DEFINITION 3.4. $\gamma(\xi) = \Phi_j(U)$.

To define $\Phi_j(U)$, following Massey [2], consider the cohomology sequence of the pair (B, E) where B replaces the mapping cylinder of p . Since $\chi(\xi) = j^*(U) = 0$, we may choose $a \in H^{n-1}(E; Z)$ such that $\delta^*(a) = U$. Since $\alpha(\xi) \equiv 0$, a may be further assumed to satisfy $\beta(a) = 0$. Then $\Phi(a)$ is defined and satisfies

$$\delta^*\Phi(a) = \Phi(\delta^*(a)) = \Phi(U) = 0 .$$

DEFINITION 3.5. $p^*(\Phi_j(U)) = \bigcup \Phi(a)$ as a ranges over elements $a \in H^{n-1}(E; Z)$ such that $\delta^*(a) = U$ and $(Sq^2 + w_2 \smile)(a) = 0$.

PROPOSITION 3.6. The indeterminacy Q of $\gamma(\xi)$ is given by

$$Q = \{ \Phi(b) + \beta(c) \} ,$$

where $b \in H^{n-1}(B; Z)$ such that $\Phi(b)$ is defined and $c \in H^*(B)$.

(3.6) and (2.1) are now evident.

4. The Peterson-Stein formula and the proof of (2.2). Twisted secondary operations satisfy the usual Peterson-Stein formulas. Stated as (4.1), for simplicity in terms of absolute cohomology classes, is the one to be used.

PROPOSITION 4.1. Let $f: Y \rightarrow X$ be a map compatible with the given structures of Y and X as spaces obtained from vector bundles. Let $x \in H^n(X; Z)$ satisfy $\beta(f^*(x)) = 0$. Then

$$\Phi(f^*(x)) \equiv \beta_f \beta(x) \in H^{n+3}(Y) \text{ mod } \beta H^{n+1}(Y) + f^* H^{n+3}(X) .$$

The proof of (4.1) is postponed to the end of this section. The functional operation β_f appearing in (4.1) is defined by the generalization of Steenrod's method as given in [7].

We now turn to the proof of (2.2). Consider the portion of the Moore-Postnikov tower for the associated sphere bundle to the universal $SO(n)$ -bundle γ_n displayed in (4.2).

Diagram 4.2.

$$\begin{array}{ccc} B_2 & \xrightarrow{k_2} & K(Z_2, n + 2) \\ \downarrow q_1 & & \\ B_1 & \xrightarrow{k_2} & K(Z_2, n + 1) \\ \downarrow q_1 & & \\ BSO(n) & \xrightarrow{\chi} & K(Z, n) . \end{array}$$

Let $\xi_1 = q_1^*(\gamma_n)$ and $\xi_2 = q_2^*(\xi_1)$. It then suffices to show $k_2 \in \gamma(\xi_2)$. By [9] $k_1 \in \alpha(\xi_1)$, while, by [1], $k_2 \in \beta_{q_2}(k_1)$.

Consider now (4.3), induced by the bundle map $q_2: \xi_2 \rightarrow \xi_1$.

Diagram 4.3.

$$\begin{array}{ccc} E_2 & \xrightarrow{p_2} & B_2 \\ \downarrow q_2 & & \downarrow q_2 \\ E_1 & \xrightarrow{p_1} & B_1 \end{array}$$

Since $k_1 \in \alpha(\xi_1)$, we may write $p_1^*(k_1) = \beta(a_1)$ for an appropriate $a_1 \in H^{n-1}(E_1)$ such that $\delta^*(a_1) = U(\xi_1)$. Let $a_2 = q_2^*(a_1)$. Then $(p_2^*)^{-1}\Phi(a_2)$ represents $\gamma(\xi_2)$.

On the other hand, since $k_2 \in \beta_{q_2}(k_1)$, by naturality

$$p_2^*(k_2) \in \beta_{q_2}(p_1^*(k_1)) = \beta_{q_2}\beta(a_1) .$$

The result follows by (4.1), which yields $\beta_{q_2}\beta(a_1) \equiv \Phi(a_2)$.

Proof of (4.1). For this proof we adopt the notations of [11]. Let $p: E, Y \rightarrow Y \times K$, Y be the universal example for Φ . Then a representative φ of $\Phi(p^*(\iota_n))$ is defined in [11] by means of a certain relative transgression sequence for p by a formula $\varphi \in \mu^{-1}\alpha\tau^{-1}\beta(\iota_n)$. However, it is proved in [12] that this transgression sequence, in the range of dimensions considered, is equivalent to the cohomology sequence of the triple (M, E, Y) , where M is the mapping cylinder of

p . Let $j: Y \times K, Y \rightarrow M, E$ be the inclusion. Translating the definition of φ to this sequence, we have $\varphi \in (\delta^*)^{-1}\beta(j^*)^{-1}\beta(\iota_n)$. But this last is precisely the definition of a representative of $\beta_p\beta(\iota_n)$. Thus (4.1) is valid in the universal example, and hence in general.

5. **Proof of (2.3).** We now consider bundles ξ and ξ' such that $\alpha(\xi)$ and $\alpha(\xi')$ are defined; let $\xi'' = \xi \oplus \xi'$. Denote by Z the mapping cylinder of p . The following is proved in [7].

PROPOSITION 5.1. There is a natural homeomorphism of pairs $Z'', E'' \rightarrow Z \times Z', E \times Z' \simeq Z \times E'$ extending the identity of $B'' = B \times B'$ and inducing a natural homeomorphism $T'' \rightarrow T'' \wedge T'$.

Now consider (5.2), in which the rows and the middle triangle are exact. The top row of (5.2) is obtained by splicing $(0 \rightarrow H^*(B) \rightarrow H^*(E) \rightarrow H^*(T) \rightarrow 0) \otimes H^*(B')$ with $H^*(T) \otimes (0 \rightarrow H^*(B') \rightarrow H^*(E') \rightarrow H^*(T') \rightarrow 0)$, while the triangle is the exact sequence of the pair $E'', E \times Z'$.

Diagram 5.2.

$$\begin{array}{ccccccc}
 0 \longrightarrow & H^*(B'') & \xrightarrow{p^* \otimes 1} & H^*(E) \otimes H^*(B') & \xrightarrow{\delta^* \otimes p'^*} & H^*(T) \otimes H^*(E') & \xrightarrow{1 \otimes \delta'^*} & H^*(T'') \longrightarrow 0 \\
 & \parallel & & \swarrow g^* & \searrow f^* & & & \parallel \\
 0 \longrightarrow & H^*(B'') & \xrightarrow{p''^*} & H^*(E'') & \xrightarrow{\delta''^*} & H^*(T'') & \longrightarrow & 0
 \end{array}$$

The proof of (2.3) is based on (5.2) as follows. Choose $a' \in H^{n-1}(E')$ such that $\delta'^*(a') = U'$. Let $a'' = f^*(U \otimes a')$. Then $\delta''^*(a'') = U''$. Further, $(Sq^2 + w_2'' \smile)(a'') = 0$, as calculation checks. Thus $(p''^*)^{-1}\Phi(a'')$ is a representative of $\gamma(\xi + \xi')$.

On the other hand, $\Phi(a'') = \Phi(f^*(U \otimes a'))$ may be evaluated by (4.1). Computing, using the Wu formula [9] $(Sq^2 + w_2 \smile)(a) = 0$ and denoting by a any class in $H^{n-1}(E; Z)$ such that $\delta^*(a) = U$, we have the following, in which $\alpha(a)$ is the representative of α determined by a .

$$\begin{aligned}
 (p''^*)^{-1}\Phi(a'') &= (p''^*)^{-1}\Phi(f^*(U \otimes a')) \\
 &= (p''^*)^{-1}\beta'_f\beta''(U \otimes a') \\
 &= (p''^*)^{-1}\beta'_f[U \otimes \beta'(a')] \\
 &= (p''^*)^{-1}(g^*)^{-1}\beta''[a \otimes \alpha'(a')] \\
 &= (p^* \otimes 1)^{-1}[\beta(a) \otimes \alpha'(a')] \\
 &= \alpha(a) \otimes \alpha'(a')
 \end{aligned}$$

modulo indeterminacies.

This completes the proof of (2.3) and in fact of the following sharpening.

COROLLARY 5.3. *Under the hypotheses of (2.3), let $\alpha(a)$ and $\alpha'(a')$ be representatives of $\alpha(\xi)$ and $\alpha(\xi')$ respectively. Then $\alpha(a) \otimes \alpha'(a')$ is a representative of $\gamma(\xi \oplus \xi')$.*

6. An example. Let $\xi + 1$ be the tangent bundle of S^{4q+1} and $\xi' + 1$ the tangent bundle of $S^{4q'+1}$ for $q, q' \geq 1$. By [9], $\alpha(\xi) \neq 0 \pmod 0$ in $H^{4q+1}(S^{4q+1})$ and similarly for ξ' . It follows by (2.3) that $\gamma(\xi \oplus \xi')$ is nonzero in $H^{4q+4q'+2}(S^{4q+1} \times S^{4q'+1})$; the indeterminacy again vanishes. Thus $\xi \oplus \xi'$ has no nonvanishing section.

This result can be obtained without the use of twisted operations, for the Whitney classes here vanish. That $\alpha(\xi) \neq 0$ reflects that Sq^2a generates $p^*H^{4q+1}(S^{4q+1})$ in $H^{4q+1}(E)$, while $\gamma(\xi + \xi') \neq 0$ reflects that $\Phi_{1,1}(a'')$ generates $p''^*H^{4q+4q'+2}(S^{4q+1} \times S^{4q'+1})$ in $H^{4q+4q'+2}(E'')$, where $\Phi_{1,1}$ is the ordinary secondary operation associated with the Adem relation $Sq^2Sq^2 = 0$, valid on integer classes.

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Received January 4, 1968. Partially supported by NSF grant GP-8955.

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The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

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