

# Pacific Journal of Mathematics

**SIMULTANEOUS INTERPOLATION IN  $H_2$ . II**

JORDAN TOBIAS ROSENBAUM

## SIMULTANEOUS INTERPOLATION IN $H_2$ , II

J. T. ROSENBAUM

Let  $\{z_n\}$  denote a fixed sequence of complex numbers in the unit disc satisfying  $(1 - |z_{n+1}|^2)/(1 - |z_n|^2) \leq \delta < 1$  for some  $\delta$ . Let  $M$  be a nonnegative integer, and let  $m$  be generic for integers between 0 and  $M$  inclusive. We define the linear functionals  $L_n^{[m]}$  on  $H_2$  by  $L_n^{[m]}f = f^{(m)}(z_n)$ . Given  $M + 1$  sequences  $w^{[0]}, \dots, w^{[M]}$  in  $l_2$ , can there be found a function  $f$  in  $H_2$  which solves the simultaneous weighted interpolation problem

$$f^{(m)}(z_n) = (w^{[m]})_n \|L_n^{[m]}\| ?$$

Shapiro and Shields considered this problem for  $M = 0$ . Their results were generalized by the author to the case  $M = 1$ . The purpose of this paper is to extend this generalization to arbitrary  $M$ .

The technique which we used for  $M = 1$  would suggest that to proceed to arbitrary  $M$ , we should let  $w^{[0]}, \dots, w^{[M]}$  be prescribed in  $l_2$  and then try to find  $f_0, \dots, f_M$  in  $H_2$  satisfying

$$(A) \quad \begin{cases} f_n^{(m)}(z_n) = (w^{[m]})_n \|L_n^{[m]}\| \\ f_n^{(i)}(z_n) = 0 \quad (0 \leq i \leq M, i \neq m) \end{cases}$$

Then,  $f_0 + \dots + f_M$  could serve as the desired interpolating function. However, the computational difficulties which would be involved in such a program can be glimpsed even in the case  $M = 1$ . We found the following modification to be effective.

The work of Shapiro and Shields assures us that we can interpolate when  $M = 0$ . Fixing  $M$  and assuming the result for lesser values, let  $w^{[0]}, \dots, w^{[M]}$  be chosen from  $l_2$ . The induction hypothesis furnishes us with a function  $f_{M-1}$  corresponding to  $w^{[0]}, \dots, w^{[M-1]}$ . We would like to alter  $f_{M-1}$  by finding a function  $g_{M-1}$  in  $H_2$  for which the sum  $f_M \equiv f_{M-1} + g_{M-1}$ , together with its first  $M$  derivatives, assumes appropriate values on  $\{z_n\}$ . This is equivalent to demanding that

$$\begin{cases} g_{M-1}^{(M)}(z_n) = [(w^{[M]})_n - \|L_n^{[M]}\|^{-1}f_{M-1}^{(M)}(z_n)] \|L_n^{[M]}\| \\ g_{M-1}^{(m)}(z_n) = 0 \quad (m < M) . \end{cases}$$

By proving that the quantity in brackets is in  $l_2$ , we reduce the problem to that of finding a function  $g$ , once  $m$  and  $w^{[m]}$  have been prescribed, which satisfies

$$(B) \quad \begin{cases} g^{(m)}(z_n) = (w^{[m]})_n \|L_n^{[m]}\| \\ g^{(i)}(z_n) = 0 \quad (i < m) . \end{cases}$$

(B) is simpler to solve than (A) because the restriction  $i \neq m$  has been changed to  $i < m$ . This accounts for why, although we now deal with arbitrary  $M$ , our work is even less computational than when we only treated the case  $M = 1$ .

## 2. Preliminary results.

2.1 In [1], Bari proved the following: Let  $\{x_n\}$  be a sequence of elements in a separable Hilbert space  $H$ . Then  $\{(x, x_n)\}$  belongs to  $l_2$  for all  $x$  in  $H$  if and only if the infinite matrix with elements  $(x_i, x_j)$  determines a bounded operator on  $l_2$ .

2.2 In [3], Schur showed that for any infinite matrix  $(a_{ij})$ , if  $\sum_i |a_{ij}| \leq N_1$  for all  $j$ , and  $\sum_j |a_{ij}| \leq N_2$  for all  $i$ , then

$$|\sum_{i,j} a_{ij} x_i \bar{x}_j| \leq (N_1 N_2)^{1/2} \sum_i |x_i|^2.$$

2.3 Let  $\delta_n$  denote  $(1 - |z_n|^2)^{-1/2}$ . We say that  $\{z_n\}$  approaches the boundary exponentially, provided that

$$\delta_n / \delta_{n+1} \leq \delta < 1 \quad (n = 1, 2, \dots)$$

for some  $\delta$ .

We say that  $\{z_n\}$  is a Carleson sequence if

$$\prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| > \sigma > 0 \quad (n = 1, 2, \dots)$$

for some  $\sigma$ .

If a sequence approaches the boundary exponentially then it is a Carleson sequence (see [4]).

2.4 The functionals  $L_n^{[m]}$  are continuous with Riesz representatives

$$K_n^{[m]}(z) = \frac{m! z^m}{(1 - \bar{z}_n z)^{m+1}}.$$

Their norms satisfy  $\delta_n^{2m+1} \leq \|L_n^{[m]}\| = O(\delta_n^{2m+1})$  (for  $M$  fixed).

This is suggested by applying  $\partial^m / \partial z_n^m$  to both sides of

$$f(z_n) = \frac{1}{2\pi i} \lim_{r \uparrow 1} \oint \frac{f(z)}{z} \frac{dz}{1 - z_n \bar{z}} \quad (|z| = r)$$

and then formally bringing the operator past the limit and the integral sign. The result is more readily established by hindsight by finding the Taylor expansion of  $m!(1 - \bar{z}_n z)^{-m-1}$  and then raising the exponents by  $m$  to get the expansion of  $K_n^{[m]}$ . The identity

$$(\Sigma a_n z^n, \Sigma b_n z^n) = \Sigma a_n \bar{b}_n$$

(for functions in  $H_2$ ) then yields

$$(f, K_n^{[m]}) = f^{(m)}(z_n) .$$

The norm can be computed easily by noting that

$$\| K_n^{[m]} \|^2 = (K_n^{[m]}, K_n^{[m]}) = \left[ \frac{d^m}{dz^m} K_n^{[m]}(z) \right]_{z=z_n} .$$

**3. Simultaneous interpolation.** We will prove that if  $\{z_n\}$  approaches the boundary exponentially, then simultaneous weighted interpolation can be done with an  $H_2$  function and its first  $M$  derivatives for  $M$  arbitrary.

**THEOREM 1.** *If  $\{z_n\}$  approaches the boundary exponentially and if  $f$  is in  $H_2$  then*

$$f^{(m)}(z_n) / \| K_n^{[m]} \|$$

is in  $l_2$  for arbitrary  $m$ .

*Proof.* By a method similar to that used for the computation of  $\| K_n^{[m]} \|$ , we find that  $|(K_n^{[m]}, K_p^{[m]})| = 0(1 - \bar{z}_n z_p)^{-2m-1}$ . Let  $k_n^{[m]}$  denote the normalization of  $K_n^{[m]}$ . Since  $1/|1 - \bar{z}_n z_p|$  is less than both  $2\delta_n^2$  and  $2\delta_p^2$  thus  $|(k_n^{[m]}, k_p^{[m]})|$  is dominated by both  $(\delta_n/\delta_p)^{2m+1}$  and  $(\delta_p/\delta_n)^{2m+1}$  and thus by  $(\delta^{2m+1})^{|n-p|}$ . This, together with Schur's result, allows us to conclude that the matrix whose elements are  $(k_n^{[m]}, k_p^{[m]})$  determines a bounded operator in  $l_2$ . Bari's theorem then applies to complete the proof.

**THEOREM 2.** *If  $\{z_n\}$  approaches the boundary exponentially and if  $M$  is any nonnegative integer then, corresponding to any choice of  $M + 1$  sequences  $w^{[0]}, \dots, w^{[M]}$  in  $l_2$ , there can be found an  $f$  in  $H_2$  for which*

$$f^{(m)}(z_n) = (w^{[m]})_n \| L_n^{[m]} \| \quad (0 \leq m \leq M; n = 1, 2, \dots) .$$

*Proof.* The proof is by induction on  $M$ . As we've noted, the case  $M = 0$  has been treated by Shapiro and Shields. Let  $M > 0$  and assume the result for lesser values. If  $w^{[0]}, \dots, w^{[M]}$  are in  $l_2$ , let  $f_{M-1}$  be a function in  $H_2$  corresponding to  $w^{[0]}, \dots, w^{[M-1]}$ . We let  $B(z)$  denote the Blaschke product for  $\{z_n\}$  and let  $B_n(z)$  denote  $B(z)$  with the factor  $\bar{z}_n(z - z_n)/z_n(1 - \bar{z}_n z)$  deleted. By Theorem 1,

$$(w')_n \equiv (w^{[M]})_n - \| L_n^{[M]} \|^{-1} f_{M-1}^{(M)}(z_n)$$

determines a sequence in  $l_2$ . Then, since  $\{z_n\}$  is a Carleson sequence,

$$(w'')_n \equiv \frac{(w')_n \prod L_n^{[M]} \prod |z_n|^M}{B_n^M(z_n) \delta_n^{2M+1} M!}$$

also determines a sequence in  $l_2$ . Again using the results of Shapiro and Shields, we can find a function  $\varphi$  in  $H_2$  for which  $\varphi(z_n) = (w'')_n \delta_n$ . We define  $f_M$  to be  $f_{M-1} + B^M \varphi$ . Clearly,  $f_M$  is in  $H_2$  and a simple computation shows that it solves our interpolation problem.

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