SIMULTANEOUS INTERPOLATION IN $H_2$. II

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Let $\{z_n\}$ denote a fixed sequence of complex numbers in the unit disc satisfying $(1 - |z_n+1|^2)/(1 - |z_n|^2) \leq \delta < 1$ for some $\delta$. Let $M$ be a nonnegative integer, and let $m$ be generic for integers between $0$ and $M$ inclusive. We define the linear functionals $L_n^{[m]}$ on $H_2$ by $L_n^{[m]}f = f^{(m)}(z_n)$. Given $M+1$ sequences $w^{[0]}, \ldots, w^{[M]}$ in $l_2$, can there be found a function $f$ in $H_2$ which solves the simultaneous weighted interpolation problem

$$f^{(m)}(z_n) = (w^{[m]})_n \| L_n^{[m]} \| \ ?$$

Shapiro and Shields considered this problem for $M = 0$. Their results were generalized by the author to the case $M = 1$. The purpose of this paper is to extend this generalization to arbitrary $M$.

The technique which we used for $M = 1$ would suggest that to proceed to arbitrary $M$, we should let $w^{[0]}, \ldots, w^{[M]}$ be prescribed in $l_2$ and then try to find $f_0, \ldots, f_M$ in $H_2$ satisfying

$$\begin{align*}
[f_m^{(i)}(z_n)] &= (w^{[m]})_n \| L_n^{[m]} \| \\
[f_m^{(i)}(z_n)] &= 0 \quad (0 \leq i \leq M, i \neq m)
\end{align*}$$

(A)

Then, $f_0 + \cdots + f_M$ could serve as the desired interpolating function. However, the computational difficulties which would be involved in such a program can be glimpsed even in the case $M = 1$. We found the following modification to be effective.

The work of Shapiro and Shields assures us that we can interpolate when $M = 0$. Fixing $M$ and assuming the result for lesser values, let $w^{[0]}, \ldots, w^{[M]}$ be chosen from $l_2$. The induction hypothesis furnishes us with a function $f_{M-1}$ corresponding to $w^{[0]}, \ldots, w^{[M-1]}$. We would like to alter $f_{M-1}$ by finding a function $g_{M-1}$ in $H_2$ for which the sum $f_M = f_{M-1} + g_{M-1}$, together with its first $M$ derivatives, assumes appropriate values on $\{z_n\}$. This is equivalent to demanding that

$$\begin{align*}
g_{M-1}^{(m)}(z_n) &= [(w^{[M]})_n - \| L_n^{[M]} \|^{-1} f_{M-1}^{(m)}(z_n)] \| L_n^{[M]} \| \\
g_{M-1}^{(m)}(z_n) &= 0 \quad (m < M) .
\end{align*}$$

By proving that the quantity in brackets is in $l_2$, we reduce the problem to that of finding a function $g$, once $m$ and $w^{[m]}$ have been prescribed, which satisfies

$$\begin{align*}
g^{(m)}(z_n) &= (w^{[m]})_n \| L_n^{[m]} \| \\
g^{(i)}(z_n) &= 0 \quad (i < m) .
\end{align*}$$

(B)

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(B) is simpler to solve than (A) because the restriction \( i \neq m \) has been changed to \( i < m \). This accounts for why, although we now deal with arbitrary \( M \), our work is even less computational than when we only treated the case \( M = 1 \).

\section{Preliminary results}

2.1 In [1], Bari proved the following: Let \( \{x_n\} \) be a sequence of elements in a separable Hilbert space \( H \). Then \( \{(x, x_n)\} \) belongs to \( l_2 \) for all \( x \) in \( H \) if and only if the infinite matrix with elements \( (x_i, x_j) \) determines a bounded operator on \( l_2 \).

2.2 In [3], Schur showed that for any infinite matrix \( (a_{ij}) \), if \( \Sigma_i |a_{ij}| \leq N_i \) for all \( j \), and \( \Sigma_j |a_{ij}| \leq N_j \) for all \( j \), then

\[ |\Sigma_i a_{ij}x_i \bar{a}_j| \leq (N_i N_j)^{1/2} \Sigma_i |x_i|^2 .\]

2.3 Let \( \delta_n \) denote \( (1 - |z_n|^2)^{-1/2} \). We say that \( \{z_n\} \) approaches the boundary exponentially, provided that

\[ \delta_n/\delta_{n+1} \leq \delta < 1 \quad (n = 1, 2, \cdots) \]

for some \( \delta \).

We say that \( \{z_n\} \) is a \textit{Carleson sequence} if

\[ \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - z_n \bar{z}_k} \right| > \sigma > 0 \quad (n = 1, 2, \cdots) \]

for some \( \sigma \).

If a sequence approaches the boundary exponentially then it is a Carleson sequence (see [4]).

2.4 The functionals \( L_n^{[m]} \) are continuous with Riesz representatives

\[ K_n^{[m]}(z) = \frac{m! z^m}{(1 - \bar{z} z)^{m+1}} .\]

Their norms satisfy \( \delta_n^{2m+1} \leq || L_n^{[m]} || = O(\delta_n^{2m+1}) \) (for \( M \) fixed).

This is suggested by applying \( \partial^m / \partial z_n^m \) to both sides of

\[ f(z_n) = \frac{1}{2\pi i} \lim_{r \downarrow 1} \int f(z) \frac{dz}{z} \frac{1}{z_n - z} \quad (|z| = r) \]

and then formally bringing the operator past the limit and the integral sign. The result is more readily established by hindsight by finding the Taylor expansion of \( m!(1 - \bar{z} z)^{-m-1} \) and then raising the exponents by \( m \) to get the expansion of \( K_n^{[m]} \). The identity
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\[(\Sigma a_n z^n, \Sigma b_n z^n) = \Sigma a_n b_n\]

(for functions in $H_2$) then yields

\[(f, K_n^{[m]}) = f^{(m)}(z_n) .\]

The norm can be computed easily by noting that

\[||K^{[m]}||^2 = (K^{[m]}_n, K^{[m]}_p) = \left[\frac{d^m}{dz^m} K^{[m]}(z)\right]_{z=z_n} .\]

3. Simultaneous interpolation. We will prove that if \(\{z_n\}\) approaches the boundary exponentially, then simultaneous weighted interpolation can be done with an $H_2$ function and its first $M$ derivatives for $M$ arbitrary.

**Theorem 1.** If \(\{z_n\}\) approaches the boundary exponentially and if $f$ is in $H_2$ then

\[f^{(m)}(z_n)/||K^{[m]}||\]

is in $l_2$ for arbitrary $m$.

**Proof.** By a method similar to that used for the computation of $||K^{[m]}||$, we find that $|(K^{[m]}_n, K^{[m]}_p)| = 0(|1 - \bar{z}_n z_p|^{-2m-1})$. Let $k^{[m]}_n$ denote the normalization of $K^{[m]}_n$. Since $1/|1 - \bar{z}_n z_p|$ is less than both $2\delta_n^2$ and $2\delta_p^2$, thus $|(k^{[m]}_n, k^{[m]}_p)|$ is dominated by both $(\delta_n/\delta_p)^{2m+1}$ and $(\delta_p/\delta_n)^{2m+1}$ and thus by $(\delta_n^{2m+1})^{n-p}$. This, together with Schur's result, allows us to conclude that the matrix whose elements are $(k^{[m]}_n, k^{[m]}_p)$ determines a bounded operator in $l_2$. Bari's theorem then applies to complete the proof.

**Theorem 2.** If \(\{z_n\}\) approaches the boundary exponentially and if $M$ is any nonnegative integer then, corresponding to any choice of $M + 1$ sequences $w^{[0]}, \ldots, w^{[M]}$ in $l_2$, there can be found an $f$ in $H_2$ for which

\[f^{(m)}(z_n) = (w^{[m]})_n ||L^{[m]}_n|| \quad (0 \leq m \leq M; n = 1, 2, \ldots) .\]

**Proof.** The proof is by induction on $M$. As we've noted, the case $M = 0$ has been treated by Shapiro and Shields. Let $M > 0$ and assume the result for lesser values. If $w^{[0]}, \ldots, w^{[M]}$ are in $l_2$, let $f_{M-1}$ be a function in $H_2$ corresponding to $w^{[0]}, \ldots, w^{[M-1]}$. We let $B(z)$ denote the Blaschke product for \(\{z_n\}\) and let $B_n(z)$ denote $B(z)$ with the factor $z_n(z - z_n)/(1 - \bar{z}_n z)$ deleted. By Theorem 1,

\[(w')_n = (w^{[M]})_n - ||L^{[M]}_n||^{-1} f^{(M)}_{M-1}(z_n) .\]
determines a sequence in $\ell_2$. Then, since $\{z_n\}$ is a Carleson sequence,

$$(w''_n) \equiv \frac{(w')_n \| L_n^M \| |z_n|^M}{B_n^v(z_n)\delta_n^{2M+1}M!}$$

also determines a sequence in $\ell_2$. Again using the results of Shapiro and Shields, we can find a function $\varphi$ in $H_2$ for which $\varphi(z_n) = (w'')_n\delta_n$. We define $f_M$ to be $f_{M-1} + B^n\varphi$. Clearly, $f_M$ is in $H_2$ and a simple computation shows that it solves our interpolation problem.

REFERENCES


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