

Pacific Journal of Mathematics

SIMULTANEOUS INTERPOLATION IN H_2 . II

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Let $\{z_n\}$ denote a fixed sequence of complex numbers in the unit disc satisfying $(1 - |z_{n+1}|^2)/(1 - |z_n|^2) \leq \delta < 1$ for some δ . Let M be a nonnegative integer, and let m be generic for integers between 0 and M inclusive. We define the linear functionals $L_n^{[m]}$ on H_2 by $L_n^{[m]}f = f^{(m)}(z_n)$. Given $M + 1$ sequences $w^{[0]}, \dots, w^{[M]}$ in l_2 , can there be found a function f in H_2 which solves the simultaneous weighted interpolation problem

$$f^{(m)}(z_n) = (w^{[m]})_n \parallel L_n^{[m]} \parallel ?$$

Shapiro and Shields considered this problem for $M = 0$. Their results were generalized by the author to the case $M = 1$. The purpose of this paper is to extend this generalization to arbitrary M .

The technique which we used for $M = 1$ would suggest that to proceed to arbitrary M , we should let $w^{[0]}, \dots, w^{[M]}$ be prescribed in l_2 and then try to find f_0, \dots, f_M in H_2 satisfying

$$(A) \quad \begin{cases} f_m^{(m)}(z_n) = (w^{[m]})_n \parallel L_n^{[m]} \parallel \\ f_m^{(i)}(z_n) = 0 \quad (0 \leq i \leq M, i \neq m) \end{cases}$$

Then, $f_0 + \dots + f_M$ could serve as the desired interpolating function. However, the computational difficulties which would be involved in such a program can be glimpsed even in the case $M = 1$. We found the following modification to be effective.

The work of Shapiro and Shields assures us that we can interpolate when $M = 0$. Fixing M and assuming the result for lesser values, let $w^{[0]}, \dots, w^{[M]}$ be chosen from l_2 . The induction hypothesis furnishes us with a function f_{M-1} corresponding to $w^{[0]}, \dots, w^{[M-1]}$. We would like to alter f_{M-1} by finding a function g_{M-1} in H_2 for which the sum $f_M \equiv f_{M-1} + g_{M-1}$, together with its first M derivatives, assumes appropriate values on $\{z_n\}$. This is equivalent to demanding that

$$\begin{cases} g_{M-1}^{(M)}(z_n) = [(w^{[M]})_n - \parallel L_n^{[M]} \parallel^{-1} f_{M-1}^{(M)}(z_n)] \parallel L_n^{[M]} \parallel \\ g_{M-1}^{(m)}(z_n) = 0 \quad (m < M) . \end{cases}$$

By proving that the quantity in brackets is in l_2 , we reduce the problem to that of finding a function g , once m and $w^{[m]}$ have been prescribed, which satisfies

$$(B) \quad \begin{cases} g^{(m)}(z_n) = (w^{[m]})_n \parallel L_n^{[m]} \parallel \\ g^{(i)}(z_n) = 0 \quad (i < m) . \end{cases}$$

(B) is simpler to solve than (A) because the restriction $i \neq m$ has been changed to $i < m$. This accounts for why, although we now deal with arbitrary M , our work is even less computational than when we only treated the case $M = 1$.

2. Preliminary results.

2.1 In [1], Bari proved the following: Let $\{x_n\}$ be a sequence of elements in a separable Hilbert space H . Then $\{(x, x_n)\}$ belongs to l_2 for all x in H if and only if the infinite matrix with elements (x_i, x_j) determines a bounded operator on l_2 .

2.2 In [3], Schur showed that for any infinite matrix (a_{ij}) , if $\sum_i |a_{ij}| \leq N_1$ for all j , and $\sum_j |a_{ij}| \leq N_2$ for all i , then

$$|\sum_{ij} a_{ij} x_i \bar{x}_j| \leq (N_1 N_2)^{1/2} \sum_i |x_i|^2.$$

2.3 Let δ_n denote $(1 - |z_n|^2)^{-1/2}$. We say that $\{z_n\}$ approaches the boundary exponentially, provided that

$$\delta_n / \delta_{n+1} \leq \delta < 1 \quad (n = 1, 2, \dots)$$

for some δ .

We say that $\{z_n\}$ is a Carleson sequence if

$$\prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| > \sigma > 0 \quad (n = 1, 2, \dots)$$

for some σ .

If a sequence approaches the boundary exponentially then it is a Carleson sequence (see [4]).

2.4 The functionals $L_n^{[m]}$ are continuous with Riesz representatives

$$K_n^{[m]}(z) = \frac{m! z^m}{(1 - \bar{z}_n z)^{m+1}}.$$

Their norms satisfy $\delta_n^{2m+1} \leq \|L_n^{[m]}\| = 0(\delta_n^{2m+1})$ (for M fixed).

This is suggested by applying $\partial^m / \partial z_n^m$ to both sides of

$$f(z_n) = \frac{1}{2\pi i} \lim_{r \uparrow 1} \oint \frac{f(z)}{z} \frac{dz}{1 - z_n \bar{z}} \quad (|z| = r)$$

and then formally bringing the operator past the limit and the integral sign. The result is more readily established by hindsight by finding the Taylor expansion of $m!(1 - \bar{z}_n z)^{-m-1}$ and then raising the exponents by m to get the expansion of $K_n^{[m]}$. The identity

$$(\Sigma a_n z^n, \Sigma \bar{b}_n z^n) = \Sigma a_n \bar{b}_n$$

(for functions in H_2) then yields

$$(f, K_n^{[m]}) = f^{(m)}(z_n) .$$

The norm can be computed easily by noting that

$$\| K_n^{[m]} \|^2 = (K_n^{[m]}, K_n^{[m]}) = \left[\frac{d^m}{dz^m} K_n^{[m]}(z) \right]_{z=z_n} .$$

3. Simultaneous interpolation. We will prove that if $\{z_n\}$ approaches the boundary exponentially, then simultaneous weighted interpolation can be done with an H_2 function and its first M derivatives for M arbitrary.

THEOREM 1. *If $\{z_n\}$ approaches the boundary exponentially and if f is in H_2 then*

$$f^{(m)}(z_n) / \| K_n^{[m]} \|$$

is in l_2 for arbitrary m .

Proof. By a method similar to that used for the computation of $\| K_n^{[m]} \|$, we find that $|(K_n^{[m]}, K_p^{[m]})| = 0(|1 - \bar{z}_n z_p|^{-2m-1})$. Let $k_n^{[m]}$ denote the normalization of $K_n^{[m]}$. Since $1/|1 - \bar{z}_n z_p|$ is less than both $2\delta_n^2$ and $2\delta_p^2$ thus $|(k_n^{[m]}, k_p^{[m]})|$ is dominated by both $(\delta_n/\delta_p)^{2m+1}$ and $(\delta_p/\delta_n)^{2m+1}$ and thus by $(\delta^{2m+1})^{|n-p|}$. This, together with Schur's result, allows us to conclude that the matrix whose elements are $(k_n^{[m]}, k_p^{[m]})$ determines a bounded operator in l_2 . Bari's theorem then applies to complete the proof.

THEOREM 2. *If $\{z_n\}$ approaches the boundary exponentially and if M is any nonnegative integer then, corresponding to any choice of $M + 1$ sequences $w^{[0]}, \dots, w^{[M]}$ in l_2 , there can be found an f in H_2 for which*

$$f^{(m)}(z_n) = (w^{[m]})_n \| L_n^{[m]} \| \quad (0 \leq m \leq M; n = 1, 2, \dots) .$$

Proof. The proof is by induction on M . As we've noted, the case $M = 0$ has been treated by Shapiro and Shields. Let $M > 0$ and assume the result for lesser values. If $w^{[0]}, \dots, w^{[M]}$ are in l_2 , let f_{M-1} be a function in H_2 corresponding to $w^{[0]}, \dots, w^{[M-1]}$. We let $B(z)$ denote the Blaschke product for $\{z_n\}$ and let $B_n(z)$ denote $B(z)$ with the factor $\bar{z}_n(z - z_n)/z_n(1 - \bar{z}_n z)$ deleted. By Theorem 1,

$$(w')_n \equiv (w^{[M]})_n - \| L_n^{[M]} \|^{-1} f_{M-1}^{(M)}(z_n)$$

determines a sequence in l_2 . Then, since $\{z_n\}$ is a Carleson sequence,

$$(w'')_n \equiv \frac{(w')_n \|L_n^{[M]}\| |z_n|^M}{B_n^M(z_n) \delta_n^{2M+1} M!}$$

also determines a sequence in l_2 . Again using the results of Shapiro and Shields, we can find a function φ in H_2 for which $\varphi(z_n) = (w'')_n \delta_n$. We define f_M to be $f_{M-1} + B^M \varphi$. Clearly, f_M is in H_2 and a simple computation shows that it solves our interpolation problem.

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Charles A. Akemann, <i>Invariant subspaces of $C(G)$</i>	421
Dan Amir and Zvi Ziegler, <i>Generalized convexity cones and their duals</i>	425
Raymond Balbes, <i>On (J, M, m)-extensions of order sums of distributive lattices</i>	441
Jan-Erik Björk, <i>Extensions of the maximal ideal space of a function algebra</i>	453
Frank Castagna, <i>Sums of automorphisms of a primary abelian group</i>	463
Theodore Seio Chihara, <i>On determinate Hamburger moment problems</i>	475
Zeev Ditzian, <i>Convolution transforms whose inversion function has complex roots in a wide angle</i>	485
Myron Goldstein, <i>On a paper of Rao</i>	497
Velmer B. Headley and Charles Andrew Swanson, <i>Oscillation criteria for elliptic equations</i>	501
John Willard Heidel, <i>Qualitative behavior of solutions of a third order nonlinear differential equation</i>	507
Alan Carleton Hindmarsh, <i>Pick's conditions and analyticity</i>	527
Bruce Ansgar Jensen and Donald Wright Miller, <i>Commutative semigroups which are almost finite</i>	533
Lynn Clifford Kurtz and Don Harrell Tucker, <i>An extended form of the mean-ergodic theorem</i>	539
S. P. Lloyd, <i>Feller boundary induced by a transition operator</i>	547
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, <i>A new proof of the maximum principle for doubly-harmonic functions</i>	567
Robert Einsohn Mosher, <i>The product formula for the third obstruction</i>	573
Sam Bernard Nadler, Jr., <i>Sequences of contractions and fixed points</i>	579
Eric Albert Nordgren, <i>Invariant subspaces of a direct sum of weighted shifts</i>	587
Fred Richman, <i>Thin abelian p-groups</i>	599
Jordan Tobias Rosenbaum, <i>Simultaneous interpolation in H_2. II</i>	607
Charles Thomas Scarborough, <i>Minimal Urysohn spaces</i>	611
Malcolm Jay Sherman, <i>Disjoint invariant subspaces</i>	619
Joel John Westman, <i>Harmonic analysis on groupoids</i>	621
William Jennings Wickless, <i>Quasi-isomorphism and TFM rings</i>	633
Minoru Hasegawa, <i>Correction to "On the convergence of resolvents of operators"</i>	641