

Pacific Journal of Mathematics

DISJOINT INVARIANT SUBSPACES

MALCOLM JAY SHERMAN

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Let $H_{\mathcal{H}}^2$ denote the (separable) Hilbert space of all functions $F(e^{i\theta})$ defined on the unit circle with values in the separable (usually infinite dimensional) Hilbert space \mathcal{H} , and which are weakly in the Hardy class H^2 . For a closed subspace of $H_{\mathcal{H}}^2$ "invariant" means invariant under the right shift operator. Such an invariant subspace is said to be of full range if it is of the form $\mathcal{U}H_{\mathcal{H}}^2$, where $\mathcal{U}(e^{i\theta})$ is a.e. a unitary operator on \mathcal{H} ; i.e., an inner function. We show that if \mathcal{H} is infinite dimensional there exists an uncountable family $\{\mathcal{M}_\alpha\}$ of invariant subspaces of $H_{\mathcal{H}}^2$ of full range such that $\mathcal{M}_\alpha \cap \mathcal{M}_\beta = (0)$ if $\alpha \neq \beta$.

This extends a theorem in the author's paper [2, p. 169] asserting the existence of *two* invariant subspaces \mathcal{M}, \mathcal{N} of full range such that $\mathcal{M} \cap \mathcal{N} = (0)$. For basic definitions and notations consult [1], particularly Chapter VI.

For a bounded operator T on \mathcal{H} , $\|T\| < 1$, define the Rota subspace \mathcal{M}_T of T to be all $F \in H_{\mathcal{H}}^2$ with Fourier series $F = \sum_{k=0}^{\infty} \varphi_k e^{ki\theta}$ such that $\sum_{k=0}^{\infty} T^k \varphi_k = 0$. It is known [2, p. 161] that \mathcal{M}_T is of full range. It was shown in [2, p. 169] that if T, U are one-to-one operators on \mathcal{H} with disjoint ranges, then $\mathcal{M}_T \cap \mathcal{M}_U = (0)$. It suffices then to prove the existence in a separable infinite dimensional Hilbert space of an uncountable family of bounded one-to-one operators with disjoint ranges. To do this it suffices to exhibit an uncountable family of disjoint *closed* infinite dimensional subspaces of a separable Hilbert space, since the subspaces are then unitarily equivalent to the original space and the operators can be taken to be of the form $U/2$, where U is unitary as a mapping onto its range. It is convenient to describe such an example in \mathcal{H} realized as $L_{\mathcal{K}}^2$, where \mathcal{K} is some other Hilbert space. Let $\{e_\alpha\}$ be an uncountable family of pairwise linearly independent vectors in \mathcal{K} (which exists if \mathcal{K} is at least two-dimensional) and for the subspaces let

$$\mathcal{N}_\alpha = \{F \in L_{\mathcal{K}}^2: F(e^{i\theta}) = f(e^{i\theta})e_\alpha, \text{ for some } f \in L^2\}.$$

The situation when \mathcal{H} is infinite dimensional thus contrasts strongly with the finite dimensional situation [1, p. 70] where the intersection of two invariant subspaces of full range also has full range, and the implication is that only when \mathcal{H} is infinite dimensional can invariant subspaces of full range be pretty small. On the

other hand, if nontrivial maximal invariant subspaces of $H^2_{\mathcal{H}}$ exist (or, equivalently, if there exist bounded operators on \mathcal{H} without nontrivial subspaces [1, p. 103]), the existence of an uncountable family of disjoint *maximal* invariant subspaces is conceivable. For if there exists an operator T on \mathcal{H} without an invariant subspace, it may also happen that T is not invertible and the codimension of the range of T is uncountably infinite in the linear space sense. It is then almost certain that one can find an uncountable family of such operators whose ranges are disjoint.

REFERENCES

1. H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.
2. M. J. Sherman, *Operators and inner functions*, Pacific J. Math. **22** (1967), 159–170.

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