HARMONIC ANALYSIS ON GROUPOIDS

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This paper generalizes harmonic analysis on groups to obtain a theory of harmonic analysis on groupoids. A system of measures is obtained for a locally compact locally trivial groupoid, \( Z \), analogous to left Haar measure for a locally compact group. Then a convolution and involution are defined on \( C_c(Z) \) = the continuous complex valued functions on \( Z \) with compact support. Strongly continuous unitary representations of \( Z \) on certain fiber bundles, called representation bundles, are lifted to \( C_c(Z) \), yielding \( \ast \) representations of \( C_c(Z) \).

A norm, \( || ||_2 \), is defined on \( C_c(Z) \), and the convolution, involution, and representations all extend to \( \mathcal{L}_2(Z) \) = the \( || ||_2 \) completion of \( C_c(Z) \). The main example given is that of the groupoid \( Z = Z(G, H) \) that arises naturally from a Lie group \( G \) and a closed subgroup \( H \). In this example, the representations of \( Z \) are related to induced representations of \( G \).

Finally, if \( Z_{ee} (= \) the group of elements in \( Z \) with left unit = right unit = \( e \) \) is compact then we canonically represent \( Z_{ee} \) as a direct sum of certain simple \( H^\ast \)-algebras.

We use extensively the notation and results of [8], except that [8] assumes a \( C^r \) manifold structure on the groupoid \( Z \), and we want to consider groupoids with just topological structure. There is no essential difficulty in developing the main results of [8] for locally trivial topological groupoids. In particular, a \( C^r \) coordinate (resp. \( C^r \) fiber) bundle in [8] becomes a coordinate (resp. fiber) bundle as defined in [7].

Reviewing [8, §1], the algebraic structure of a (transitive) groupoid, \( Z \) (over \( M \)), consists of a subset \( M \) of \( Z \) (called the units of \( Z \)), a projection \( l \times r \) of \( Z \) onto \( M \times M \) sending \( \Phi_{qp} \in Z \) into (left unit \( \Phi_{qp} \), right unit of \( \Phi_{qp} \) = \( q, p \)), and a law of composition defined for pairs \( \Phi_{qp}, \Phi_{rs} \) such that \( p = r \). For \( B \subseteq M \times M, Z_B \) is defined as \( (l \times r)^{-1}(B) \), and \( Z_{qp} = (l \times r)^{-1}(q, p) \). The composition \( \Phi_{qp} \cdot \Phi_{ps} \in Z_{qs} \), and \( (\Phi_{qp}, \Phi_{ps}) \cdot \Gamma_{st} = \Phi_{qs}(\Gamma_{ps}, \Gamma_{st}) \). The unit \( q \in M \) may be written \( 1_{qq} \), and \( \Phi_{qp} = \Phi_{qp} \cdot 1_{pp} = \Phi_{pp} \). Also, \( \Phi_{qp} \) has an inverse, \( \Phi_{qp}^{-1} \), such that \( \Phi_{qp}^{-1} \cdot \Phi_{qp} = 1_{pp} \) and \( \Phi_{qp} \cdot \Phi_{qp}^{-1} = 1_{qq} \).

A coordinate groupoid \((Z, \Sigma_e)\) over \( M \) consists of the following:

1. An (algebraic, transitive) groupoid \( Z \) over \( M \) and a Hausdorff topological structure for \( M \).
2. A distinguished point \( e \in M \) and a Hausdorff topological group structure for the group \( Z_{ee} \).
3. A set of functions \( \Sigma_e = \{ \alpha: U_\alpha \rightarrow Z_{U_\alpha \times e} \} \) such that \( U_\alpha \) is
open in $M$ and $l \cdot \alpha = \text{identity map}$, satisfying

\[(1.3.1) \bigcup_{\alpha \in \Sigma_e} U_{\alpha} = M .\]

\[(1.3.2) \text{For } \alpha \text{ and } \beta \in \Sigma_e, \text{ the map } g_{\alpha \beta} : U_{\alpha} \cap U_{\beta} \to Z_{ee}; g_{\alpha \beta}(q) = \alpha(q)^{-1} \circ \beta(q), \text{ is continuous.}\]

Then the constructions of [8] lead to a topological structure for $Z$, making $Z$ a locally trivial topological groupoid as defined by Ehresmann in [3]. Conversely, any such groupoid arises from a coordinate groupoid.

Finally, we stipulate that the letter "$Z$" will always represent a locally compact locally trivial groupoid. Note $Z$ is locally compact if and only if both $Z_{ee}$ and $M$ are locally compact.

2. We first consider systems of measures on a groupoid, $Z$ over $M$.

**Definition 2.1.** A (continuous) system of measures on $Z$ is an indexed set $\lambda = \{\lambda_{qp}(q, p) : (q, p) \in M \times M\}$, where $\lambda_{qp}$ is a regular Borel measure on $Z_{qp}$. We will write $\lambda_{qp}(f) = \int_{Z_{qp}} f(\phi_{qp}) d\lambda_{qp}$, where $f$ is an integrable function on $Z_{qp}$, and will require that the function $\lambda(h) : M \times M \to C; \lambda(h)(q, p) = \lambda_{qp}(h \mid _{Z_{qp}})$ be in $C_c(M \times M)$ whenever $h \in C_c(Z)$.

The concepts of "left and right invariance" are easily applied to systems of measures.

**Definition 2.2.** A system of measures, $\lambda$, is said to be left invariant if and only if

\[
\int_{Z_{qp}} f(\psi_{qr} \cdot \phi_{rp}) d\lambda_{rp} = \int_{Z_{qp}} f(\Gamma_{qr}) d\lambda_{qp},
\]

for all $\psi_{qr} \in Z$ and $p \in M$ and $f \in C_c(Z_{qp})$. Similarly, for right invariance the condition is (with $f \in C_c(Z_{pr})$):

\[
\int_{Z_{pq}} f(\phi_{pq} \cdot \psi_{qr}) d\lambda_{pq} = \int_{Z_{pr}} f(\Gamma_{pr}) d\lambda_{pr}.
\]

If $Z_{ee}$ is unimodular, it is easy to obtain a left and right invariant system of measures for $Z$ from a Haar measure on $Z_{ee}$ (use (2.6.1) with $\Delta = 1$). In the general case, we extend the modular function for $Z_{ee}$ to $Z$, and then obtain a left invariant system of measures for $Z$ (depending on the extension).
DEFINITION 2.3. A function $A: \mathbb{Z} \rightarrow \mathbb{R}^+$ is called a modular function for $\mathbb{Z}$ if and only if:

(2.3.1) $A$ is a continuous homomorphism (multiplicative structure for $\mathbb{R}^+ =$ real numbers $> 0$.)

(2.3.2) $A|_{\mathbb{Z}_{ee}}$ is the modular function for $\mathbb{Z}_{ee}$.

THEOREM 2.4. If $M$ is paracompact, then there exists a modular function for $\mathbb{Z}$. Given two modular functions, $\Delta$ and $\Delta'$, on $\mathbb{Z}$, we have $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{\mathbb{Z}ee})$, $h: M \times M \rightarrow \mathbb{R}^+$ is a continuous homomorphism (with the trivial groupoid structure on $M \times M$, see (3.5b)).

Proof. Let $\Sigma_\alpha$ be a set of local sections in $\mathbb{Z}_{\mathbb{Z}M \times \beta}$ such that $\{U_\alpha = \text{dom } \alpha: \alpha \in \Sigma_\alpha\}$ is a locally finite cover of $M$ (using the paracompactness of $M$) and let $\{f_\alpha\}$ be a partition of 1 such that support $(f_\alpha) \subseteq U_\alpha$. $\Delta_{ee}$ is the modular function for $\mathbb{Z}_{ee}$. We define $\Delta = e^\delta$, where

$$\delta(\Phi_{qp}) = \sum_{\alpha, \beta} f_\alpha(q)f_\beta(p) \log (\Delta_{ee}(\alpha(q)^{-1} \cdot \Phi_{qp} \cdot \beta(p))).$$

Then $\Delta$ is a modular function for $\mathbb{Z}$. Given a continuous homomorphism $h: M \times M \rightarrow \mathbb{R}^+$, then $\Delta'$ defined by $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{\mathbb{Z}ee})$ is a modular function for $\mathbb{Z}$. Conversely, given two modular functions $\Delta$ and $\Delta'$ on $\mathbb{Z}$, we find that $h(q, p) = \Delta'(\Phi_{qp})/\Delta(\Phi_{\mathbb{Z}ee})$ is independent of $\Phi_{qp}$ for the given units, and that $h: M \times M \rightarrow \mathbb{R}^+$ is a continuous homomorphism.

THEOREM 2.5. If $\lambda$ is a left (resp. right) invariant system of measures on $\mathbb{Z}$, then $\lambda_{ee}$ is a left (resp. right) Haar measure on $\mathbb{Z}_{ee}$ for each $q \in M$.

From here on we assume $\lambda_{ee}$ is a fixed left Haar measure on $\mathbb{Z}_{ee}$, and will write $\lambda_{ee}(f) = \int_{\mathbb{Z}_{ee}} f(\Phi_{ee})d\Phi_{ee}$.

THEOREM 2.6. There is a natural one-to-one correspondence between the left invariant systems of measures on $\mathbb{Z}$ and the modular functions on $\mathbb{Z}$.

Proof. Given a modular function, $\Delta$, on $\mathbb{Z}$, we define the system of measures, $\lambda$, by

$$\lambda_{qp}(f) = \int_{\mathbb{Z}_{qp}} f(\Phi_{\mathbb{Z}ee})d\Phi_{\mathbb{Z}ee}$$

(2.6.1)

$$= \int_{\mathbb{Z}_{ee}} \Delta(\Gamma_{\mathbb{Z}ee})f(\mathbb{F}_{qee}\Gamma_{ee})d\Lambda_{ee}.$$
\( \lambda_{v_e} \) is independent of the choice of \( \Psi_{v_e} \) and \( \Gamma_{v_e} \) with the indicated units, and \( \lambda \) is left invariant. Conversely if \( \lambda \) is a left invariant system of measures the above equation defines \( \Delta \) on \( Z_{e \times M} \). Then \( \Delta \) may be extended to a continuous homomorphism of \( Z \) into \( R^+ \), and \( \Delta |_{Z_{ee}} \) is the modular function of \( Z_{ee} \).

**Theorem 2.7.** If \( Z_{ee} \) is unimodular, then there is a unique left and right system of measures on \( Z \) (recall \( \lambda_{ee} \) is a fixed left Haar measure).

**Proof.** Just choose \( \Delta = 1 \).

From here on we will assume that a fixed modular function \( \Delta \) has been given for \( Z \), and the corresponding left invariant system of measures is \( \lambda \) as defined in (2.6.1). A fixed regular Borel measure, \( \mu \), is specified for \( M \), and \( \mu(f) \) will be written \( \int_M f(q) dq \), for any integrable function \( f \) on \( M \). We require support of \( \mu = M \).

3. **Definition 3.1.** Given \( f \) and \( g \in C_c(Z) \) we define the convolution of \( f \) and \( g \), by \( f \ast g(\Phi_{v_e}) = \int_{\mathbb{M}} \int_{\mathbb{Z} \times V} f(\Psi_{v_r}) g(\Psi_{v_r}^{-1} \cdot \Phi_{v_e}) d\Psi_{v_r} dr \).

**Theorem 3.2.** \( C_c(Z) \) forms an algebra over \( C \) with convolution as the law of multiplication, and the usual addition and scalar multiplication.

**Proof.** The main points to verify are:

(a) \( f \ast g \in C_c(Z) \) and

(b) \( (f \ast g) \ast h = f \ast (g \ast h) \).

In regard to (a), if support \( (f) \subseteq A \) and support \( (g) \subseteq B \), then it is easy to show that support \( (f \ast g) \subseteq A \cdot B \). \( A \cdot B \) is the image of \( (A \times B) \cap D \subseteq Z \times Z \) under composition, where \( D \) is the (closed) subset of \( Z \times Z \) where composition is defined. Hence \( A \cdot B \) is compact if \( A \) and \( B \) are compact.

In regard to (b), we compute \( (f \ast g) \ast h(\Phi_{v_e}) \)

\[
= \int_{\mathbb{M}} \int_{\mathbb{Z} \times V} f(\Psi_{v_r}) g(\Psi_{v_r}^{-1} \cdot \Gamma_{v_e}) d\Psi_{v_r} dr \int_{\mathbb{M}} \int_{\mathbb{Z} \times V} h(\Gamma_{v_e}^{-1} \cdot \Phi_{v_e}) d\Gamma_{v_e} ds.
\]

Substitute \( A_{v_e} = \Psi_{v_r}^{-1} \cdot \Gamma_{v_e} \), and interchange the order of integration to obtain

\[
= \int_{\mathbb{M}} \int_{\mathbb{Z} \times V} f(\Psi_{v_r}) \left( \int_{\mathbb{M}} \int_{\mathbb{Z} \times V} g(A_{v_e}) h(A_{v_e}^{-1} \cdot \Psi_{v_r}^{-1} \cdot \Phi_{v_e}) dA_{v_e} ds \right) d\Psi_{v_r} dr
\]

\[
= f \ast (g \ast h)(\Phi_{v_e}).
\]

Next, we define an involution for \( C_c(Z) \).
DEFINITION 3.3. Given $f \in C_c(Z)$, we define $f^*$ by

$$f^*(\Phi_{qp}) = \overline{f(\Phi_{qp}^{-1})}\Delta(\Phi_{qp}^{-1})$$

(where $\overline{f}$ is the complex conjugate of $f$).

THEOREM 3.4. The map $f \mapsto f^* : C_c(Z) \to C_c(Z)$ is an involution (see [6]).

Proof. The only difficult part is to show $(f*g)^* = g^*f^*$. We compute

$$\int_{H} \int_{Z_{pr}} \overline{f(\Psi_{pr})}\overline{g(\Psi_{pr}^{-1})}\Delta(\Phi_{qp}^{-1})d\Psi_{pr}dr$$

(substituting $\Gamma_{qr} = \Phi_{qp}\Psi_{pr}$)

$$\int_{H} \int_{Z_{qr}} \overline{g(\Gamma_{qr}^{-1})}\overline{f(\Phi_{qp}^{-1})}\Delta(\Phi_{qp}^{-1})d\Gamma_{qr}dr = (g^*f^*)(\Phi_{qp}).$$

EXAMPLES 3.5. (a) Suppose $M = \{e\}$ and $\mu(1) = 1$. Then $Z = Z_{es}$ is a locally compact group, $f*g$ is the ordinary convolution, and $f \mapsto f^*$ is the usual involution.

(b) Suppose $Z = M' \times M'$ and $M = \text{diagonal of } M' \times M'$. We define the trivial groupoid structure for $Z$ over $M$ as follows:

$$l(q, p) = (q, q) \quad \text{and} \quad r(q, p) = (p, p),$$

composition is given by $(q, p) \cdot (p, r) = (q, r)$, and $(q, q) \to (q, e)$ gives a global section of $l: Z_{M \times e} \to M$.

If $M'$ is discrete, then $f^*g$ is the matrix composition of $f$ and $g$.

(c) Suppose $G$ is a Lie group and $H$ is a closed subgroup of $G$. We define the homogeneous space groupoid for $G$ and $H$, $Z(G, H) = Z = \{(q, \Phi, p) : \Phi \in G, p \in G/H, \text{ and } \Phi p = q\}$. The groupoid structure for $Z$ is given as follows: $M = \{(q, 1, q) : q \in G/H\}$ is the set of units, and $q \to (q, 1, q)$ identifies $M$ with $G/H$ to give $M$ the required topology; $l(q, \Phi, p) = (q, 1, q)$ and $r(q, \Phi, p) = (p, 1, p)$. Composition is defined by $(q, \Phi, p) \cdot (p, \Psi, r) = (q, \Phi \cdot \Psi, r)$; the local sections of $l: Z_{M \times e} \to M$ come from local sections of $G \to G/H$ (identifying $G/H$ with $M$ as above, and taking $e = 1H$); $(e, \Phi, e) \to \Phi$ is a group isomorphism sending $Z_{es}$ onto $H$, giving $Z_{es}$ the required topology.

We note that $Z_{M \times e}$ is essentially the usual principal bundle obtained from $G$ and $H$.

For simplicity we only consider in this paper the case where $\Delta_H$ (the modular function for $H$) = $\Delta_G$ (the modular function for $G$), restricted to $H$. Then, by a theorem in [5, Chapter 10], there is a $G$
invariant measure on $M$, which we take for $\mu$. There is a canonical (continuous) homomorphism $\zeta: \mathbb{Z} \to G$, defined by $\zeta(q, \Phi, p) = \Phi$. Note that $\zeta$ maps $\mathbb{Z}$ onto $G$, and that $\zeta |_{\mathbb{Z}}$ is an isomorphism mapping $\mathbb{Z}$ onto $H$. The above consideration leads to the following:

**Theorem 3.5.1.** $\Delta_0 \cdot \zeta$ is a modular function for $\mathbb{Z}$. Unless otherwise mentioned we will always use $\Delta = \Delta_0 \cdot \zeta$ for $Z(G, H)$.

If $M$ is compact and $\mu(1) = 1$, then $\xi(f) = f \cdot \zeta \in C_\Delta(Z)$ for $f \in C_\Delta(Z)$, and we obtain the

**Theorem 3.5.2.** $\zeta^*: C_\Delta(G) \to C_\Delta(Z)$ is a one-to-one* homomorphism (with the usual convolution and involution on $C_\Delta(G)$, using a suitable left Haar measure on $G$).

**Proof.** The first point is that $f \mapsto \int_M \int_{\mathbb{Z}} \xi^*(f)(\Phi q, p) d\Phi q dp$ (writing $(q, \Phi, p) = \Phi q$) defines a left invariant measure on $G$ which we take as the desired left Haar measure on $G$. Note, this measure on $G$ is independent of the choice of $q \in M$. Next, we compute

$$\xi^*(f \ast \xi^*(g))(\Phi q) = \int_{\mathbb{Z}} \xi^*(f)(\Phi q, r) \xi^*(g)(\Phi q^{-1}, \Phi q r) d\Phi q d\Phi r$$

$$= \int_G f(\Phi q) g(\Phi q^{-1}, \Phi q r) d\Phi q$$

$$= (f \ast g)(\Phi q) = \xi^*(f \ast g)(\Phi q),$$

as required.

Finally, for $f \in C_\Delta(G)$,

$$(\xi^*(f))^*(\Phi q) = (\xi^*(f))(\Phi q^{-1})^* \Delta(\Phi q^{-1}) = f(\Delta^{-1}) \Delta(\Phi q^{-1}) = \xi^*(f^*)(\Phi q),$$

as required.

4. **Definition 4.1.** A (unitary) representation bundle, $E$, is a fiber bundle with a Hilbert space structure for the fiber $Y$, and group $U(Y) = \text{the unitary operators on } Y$ with the strong operator topology.

We note that there is a natural inner product field, $\langle \cdot , \cdot \rangle$, on $E$. For $q \in M$, $\langle \cdot , \cdot \rangle_q$ is an inner product on $E_q$ defined via any admissible map from the fiber $Y$. Then $\langle \cdot , \cdot \rangle_q$ makes $E_q$ a Hilbert space and the unitary maps from $Y$ to $E_q$ are the admissible maps from $Y$ to $E_q$.

Using the given regular Borel measure, $\mu$, on $M$, we obtain an inner product on $\Gamma_\Delta(E)$, the continuous sections in $E$ with compact support. For $\gamma$ and $\delta \in \Gamma_\Delta(E)$,

$$\langle \gamma , \delta \rangle = \int_M \langle \gamma q, \delta q \rangle_d q.$$
The completion of \( \Gamma_c(E) \) with respect to this inner product is then a Hilbert space, to be called \( \Gamma_c'(E) \).

**Definition 4.2.** A (strongly continuous) unitary representation \( \rho \) of \( Z \) on a representation bundle \( E \) is a continuous homomorphism \( \rho: Z \to A(E) = \) the (locally trivial) groupoid of admissible maps between the fibers of \( E \), such that \( \rho \) is the identity map on the units of \( Z \) (see [8]).

The main results listed below are obtained essentially as in [8, §4].

**Theorems 4.3.** (a) If \( \rho \) is given as in (4.2) then \( \rho \mid _{z_{ee}} = \rho_e \) defines a unitary representation of \( Z_{ee} \) on \( E_e \).

(b) Given a unitary representation \( \rho_e \) of \( Z_{ee} \) on a Hilbert space \( E_e \), there is a representation bundle \( E' \) and representation \( \rho' \) of \( Z \) on \( E' \) such that \( \rho' \mid _{z_{ee}} \cong \rho_e \) (a unitary equivalence).

(c) Two representations \( \rho \) and \( \rho' \) of \( Z \) on \( E \) and \( E' \) respectively are equivalent (as in [8]) if and only if \( \rho \mid _{z_{ee}} \cong \rho' \mid _{z_{ee}} \).

A groupoid representation, \( \rho \), of \( Z \) on \( E^p \) defines a representation of the algebra \( C_c(Z) ; \rho: C_c(Z) \to \mathcal{L}(\Gamma_c'(E^p)) = \) the bounded linear maps of \( \Gamma_c'(E^p) \) into itself.

**Definition 4.4.** Given \( f \in C_c(Z) \) and \( \gamma \in \Gamma_c'(E^p) \), we define \( \rho(f)\gamma \) by \( (\rho(f)\gamma)_q = \int \int f(\Phi_{qp})\rho(\Phi_{qp})\gamma_p \, dq \, dp \). Alternatively,

\[
\langle \rho(f)\gamma, \delta \rangle = \int \int \int f(\Phi_{qp})\langle \rho(\Phi_{qp})\gamma_p, \delta_q \rangle \, dq \, dp \, d\Phi_{qp}.
\]

**Theorem 4.5.** \[ || \rho(f)\gamma ||_2 \leq || f ||_2 || \gamma ||_2 , \] where

\[
|| f ||_2 = \int \int \left( \int_{z_{ee}} |f(\Phi_{qp})| \, dq \, dp \right)^2 \, d\Phi_{qp}.
\]

**Proof.** See (5.4). Accordingly \( \rho(f) \) extends to a bounded operator on \( \Gamma_c'(E^p) \) of norm \( \leq || f ||_2 \). \( \mathcal{L}(\Gamma_c'(E^p)) \) has a natural Banach* algebra structure.

**Theorem 4.6.** The representation \( \rho: C_c(Z) \to \mathcal{L}(\Gamma_c'(E^p)) \) is a *homomorphism.

**Proof.** For \( f \) and \( g \in C_c(Z) \), we compute
\[(\rho(f * g)g)_{q} = \int_{M} \int_{x} f(\Phi_{q})g(\Phi_{q})d\Phi_{q}d\rho \]
\[= (\text{substituting } \Gamma_{r} = \Phi_{q}^{-1} \text{ and interchanging the order of integration}) \]
\[\int_{M} \int_{x} f(\Phi_{q})\rho(\Phi_{q}) \left( \int_{M} \int_{x} g(\Gamma_{r})\rho(\Gamma_{r})d\Gamma_{r}d\rho \right) d\Phi_{q}d\rho \]
\[= \langle \rho(f)(\rho(g)), g \rangle_{q} \text{ as desired.} \]

Finally, we compute

\[\langle \rho(f^*)\gamma, \delta \rangle = \int_{M} \int_{x} f^*(\Phi_{q})\rho(\Phi_{q}) \left( \int_{M} \int_{x} \gamma(\Phi_{q})d\Phi_{q}d\rho \right) dq \]
\[= \int_{M} \int_{x} f(\Phi_{q})^{-1}(\Delta(\Phi_{q})^{-1}) \left( \int_{M} \int_{x} \gamma(\Phi_{q})d\Phi_{q}d\rho \right) dq \]
\[(\text{see (5.2.1))} \quad \int_{M} \int_{x} f(\Phi_{q})^{-1}(\Delta(\Phi_{q})^{-1}) \left( \int_{M} \int_{x} \gamma(\Phi_{q})d\Phi_{q}d\rho \right) dq \]
\[= \langle \gamma, \rho(f)\delta \rangle, \text{ so } \rho(f^*) = \rho(f)^*. \]

The following example provides a representation analogous to the left regular representation for groups.

**Example 4.7.** Let \(\rho_{e}\) be the strongly continuous unitary representation of \(Z_{e}\) on \(L_{2}(Z_{e} \times M)\) given by \((\rho_{e}(\Phi_{e}))f_{e})(\Phi_{e}) = f_{e}(\Phi_{e}^{-1})\cdot \Phi_{e}\). The representation bundle \(F\) arising from \(\rho_{e}\) and \(Z\) may be regarded as \(= \bigcup_{q \in Z_{e}} L_{2}(Z_{e} \times M)\). The map \(f \rightarrow f' ; C_{e}(Z) \rightarrow \Gamma_{e}(F)\), defined by \(f'(q) = f |_{q \times M}\) is bijective, and \(\|f\|_{2} = \|f'\|_{2}\). Accordingly, we can identify \(L_{2}(Z)\) and \(\Gamma_{e}(F)\). Given \(f\) and \(g \in C_{e}(Z)\), then \(\rho(f)g' = (f * g)'\).

**5. Definition 5.1.** For \(f \in C_{e}(Z)\), we define

\[\|f\|_{2} = \left( \int_{M} \int_{Z_{e}} \left( \int_{Z_{e}} | f(\Phi_{q})|^2 \right) dq d\Phi_{q} \right)^{1/2}. \]

\(\|\|_{2}\) defines a norm on \(C_{e}(Z)\); we complete \(C_{e}(Z)\) with respect to \(\|\|_{2}\) to form \(L_{2}(Z)\).

To simplify matters, we recall the map: \(\lambda : C_{e}(Z) \rightarrow C_{e}(M \times M)\), where \(\lambda(f)(q, p) = \int_{Z_{e}} f(\Phi_{q})d\Phi_{q}\).

**Theorem 5.2.** \(\lambda(f * g) = \lambda(f)^{*}\lambda(g)\) and \(\lambda(f^*) = \lambda(f)^{*}\), using the trivial groupoid structure on \(M \times M\) over the diagonal of \(M \times M\). (on \((M \times M)_{e} = \{(e, e)\}\) the Haar measure is taken as 1).
Proof. We write \( f_{qp} \) for \( \lambda(f)(q, p) \). Then

\[
\lambda(f * g)(q, p) = \int_{\mathcal{Z}} \int_{\mathcal{Z}} f(\mathcal{W}_{qr})g(\mathcal{W}_{qp}^{-1} \cdot \phi_{pq})d\mathcal{W}_{qr}d\phi_{pq}
\]

\[
= \int_{\mathcal{M}} \int_{\mathcal{Z}} f(\mathcal{W}_{qr})g_{qr}d\mathcal{W}_{qr}d\rho
\]

\[
= \int_{\mathcal{M}} f_{qr}g_{qr}d\rho = (\lambda(f) \ast \lambda(g))(\phi_{pq})
\]

Next, to show \( \lambda(f^*) = \lambda(f)^* \) we should show

(5.2.1) \[
\int_{\mathcal{Z}} f(\phi_{pq}^{-1})d(\phi_{pq}^{-1})d\phi_{pq} = \int_{\mathcal{Z}} f(\phi_{pq})d\phi_{pq}.
\]

If \( p = q = e \) this is a standard theorem. The extension to the general case is routine, using (2.6.1).

Accordingly, \( f \mapsto \lambda(f) \) defines a *homomorphism. Also, \( ||f||_2 = ||\lambda(|f|)||_2 \), where \( || \cdot ||_2 \) is the \( \mathcal{L}_2 \) norm on \( C_c(M \times M) \). For \( f \) and \( g \in C_c(M \times M) \) it is easy to show that \( ||f \ast g||_2 \leq ||f||_2 ||g||_2 \). Finally, we obtain the

**Theorem 5.3.** Given \( f \) and \( g \in C_c(Z) \) then \( ||f \ast g||_2 \leq ||f||_2 ||g||_2 \) and \( ||f|| = ||f^*|| \).

**Proof.**

\[
||\lambda(|f \ast g|)||_2 \leq ||\lambda(|f| \ast |g|)||_2 = ||\lambda(|f|) \ast \lambda(|g|)||_2 \leq ||f||_2 ||g||_2
\]

settles the first part, and \( ||\lambda(|f^*|)||_2 = ||\lambda(|f|)^*||_2 = ||\lambda(|f|)||_2 ||f||_2 \) settles the second part.

Accordingly, the convolution and (*) involution extend to \( \mathcal{L}_{12}(Z) \), making \( \mathcal{L}_{12}(Z) \) a Banach algebra with a natural involution. Representations also extend to \( \mathcal{L}_{12}(Z) \) as shown below.

**Theorem 5.4.** For \( f \in C_c(Z) \) and \( \gamma \in \Gamma_c(E) \), \( ||\rho(f)\gamma||_2 \leq ||f||_2 ||\gamma||_2 \).

**Proof.**

\[
\langle \rho(f)\gamma, \rho(f)\gamma \rangle
\]

\[
= \int_{\mathcal{M}} \int_{\mathcal{M}} \int_{\mathcal{Z}} \int_{\mathcal{Z}} f(\phi_{qp})\bar{f}(\mathcal{W}_{qr})\langle \rho(\phi_{pq})\gamma_p, \rho(\mathcal{W}_{qr})\gamma_r \rangle d\mathcal{W}_{qr}d\mathcal{W}_{pq}d\rho dq
\]

\[
\leq \int_{\mathcal{M}} \int_{\mathcal{M}} ||f_{qp}|| ||\gamma_p|| ||f_{qr}|| ||\gamma_r|| d\rho dq
\]

\[
= \int_{\mathcal{M}} \left( \int_{\mathcal{M}} ||f_{qp}|| ||\gamma_p|| dp \right) \left( \int_{\mathcal{M}} ||f_{qr}|| ||\gamma_r|| dr \right) dq
\]

\[
\leq \int_{\mathcal{M}} \left( \int_{\mathcal{M}} ||f_{qp}|| ||\gamma_p||^2 dp \right) dq
\]
Accordingly, $\rho$ of $Z$ on $E$ lifts to a $*$-representation of $S^\omega(Z)$ on $\Gamma_2(E)$.

**Example 5.5.** Suppose $Z = Z(G, H)$ as in (3.5c), and that $G/H$ is compact and $\mu(1) = 1$. Then $\zeta^*: C_c(G) \to C_c(Z)$ (see (3.5.2) is a norm increasing $*$-homomorphism.

Furthermore, a representation $\rho$ of $Z$ on $E$ defines a representation $\rho'$ of $G$ on $\Gamma_2(E)$, by $(\rho'(\Phi)\gamma)_g = \rho(\Phi_{gg})\gamma_g$, where $p = \Phi^{-1}(q)$ and $\Phi_{gg} = (q, \Phi, p)$. $\rho'$ is a unitary representation since $\mu$ is invariant under $G$. Then $\rho'$ is the induced representation (well known in group theory) from the representation $\rho_e$ of $Z_e(\cong H)$ on $E_e$. The diagram below, relating $Z$ and $G$, commutes.

$$
\begin{array}{ccc}
C_c(Z) & \xrightarrow{\rho} & \mathcal{L}(\Gamma_2(E)) \\
\zeta^* & & \\
C_c(G) & \xrightarrow{\rho'} & \mathcal{L}(\Gamma_2(E')).
\end{array}
$$

Note that the case $H = G$, $\mu(1) = \lambda_e(1) = 1$, is the same as the Example 3.5a, where $Z = Z_{ee}$.

6. Suppose $Z_{ee}$ is compact, $\Delta = 1$, and $\lambda_e(1) = 1$ (the vertically compact case). Then the completion of $C_c(Z)$ with respect to the $\| \cdot \|_2$ norm forms the Hilbert space $\mathcal{L}_2(Z)$. We will extend the “orthogonality relations” for compact groups to the above case, and represent $\mathcal{L}_2(Z)$ as a direct sum of simple $H^*$ algebras.

**Definition 6.1.** Given $\gamma$ and $\delta \in \Gamma_2(E^\rho)$, where $\rho$ is a representation of $Z$ on $E^\rho$, we define $T_{\rho_{\gamma\delta}}: Z \to C$, by

$$
T_{\rho_{\gamma\delta}}(\Phi_{gg}) = \langle \gamma_q, \rho(\Phi_{gg})\delta_g \rangle_q.
$$

**Theorem 6.2.** If $\rho_e$ and $\rho'_e$ are irreducible, then

$$
\langle T_{\rho_{\gamma\delta}} T_{\rho'_{\gamma'\delta'}} \rangle = \begin{cases} 
\dim \rho_e & \text{if } \rho = \rho' \\
0 & \text{if } \rho \text{ is not equivalent to } \rho'.
\end{cases}
$$

**Proof.** Integrating both sides of (6.2.1) over $M \times M$ yields the desired result.
(6.2.1) \[
\int_{\varepsilon_{qp}} \langle \gamma, \rho(\Phi_q) \delta \rangle \frac{d\Phi_q}{\dim \rho_e} \langle \gamma', \rho'(\Phi_q') \delta' \rangle \frac{d\Phi_{q'}}{\dim \rho_{e'}} = \begin{cases}
\langle \gamma, \gamma' \rangle \frac{\dim \rho_e}{\dim \rho_{e'}} & \text{if } \rho = \rho' \\
0 & \text{if } \rho \text{ is not equivalent to } \rho'.
\end{cases}
\]

For \( q = p = e \), (6.2.1) is just the orthogonality relations for compact groups. The proof of (6.2.1) for general \( p \) and \( q \) is similar to the usual derivation of the orthogonality relations, for example see [1].

**Notation.** The representations \( \rho \) and \( \rho' \) of \( Z \) on \( E^\rho \) and \( E'^{\rho'} \) respectively will be such that \( \rho_e \) and \( \rho_{e'} \) are irreducible. The map \( \delta \rightarrow \delta^* \colon \Gamma_e(E) \rightarrow \Gamma_e(E)^* = \text{dual of } \Gamma_e(E) \), is defined by \( \delta^*(\gamma) = \langle \gamma, \delta \rangle \). \( \Gamma_e(E)^* \) is the image of \( \Gamma_e(E) \) under \( \delta \rightarrow \delta^* \). The (algebraic) tensor product \( \Gamma_e(E^\rho) \otimes \Gamma_e(E'^{\rho'}) \) many be regarded as a (dense) subalgebra of \( C_\rho \) = the Schmidt operators on \( \Gamma_e(E^\rho) \). In particular \( \langle \gamma \otimes \delta^*(\beta) \rangle = \langle \beta, \delta \gamma \rangle \). Conversely, \( \alpha \) and \( \beta \in C_\rho \) can be regarded as elements of the (Hilbert space) tensor product \( \Gamma_e(E^\rho) \otimes \Gamma_e(E'^{\rho'}) \). The inner product on \( C_\rho \) is defined by \( \langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle \text{dim } \rho_e \) where \( \langle , \rangle \) is the inner product on \( \Gamma_e(E^\rho) \otimes \Gamma_e(E'^{\rho'}) \), making \( C_\rho \) a simple \( H^* \) algebra.

**Theorem 6.4.** The canonical map \( T_\rho \colon \Gamma_e(E^\rho) \otimes \Gamma_e(E'^{\rho'}) \rightarrow C_\rho(Z) \) defined by \( T_\rho(\gamma \otimes \delta^*) = T_{\rho\gamma} \text{dim } \rho_e \) extends to a \(*\) homomorphism and isometry of \( C_\rho \) into \( \mathcal{L}(Z) \).

**Proof.** To show \( T_\rho \) defines an isometry from \( C_\rho \) we compute
\[
\langle T_{\rho\gamma} \text{dim } \rho_e, T_{\rho\gamma'} \text{dim } \rho_e \rangle = \langle \gamma \otimes \delta^* \gamma', \gamma' \otimes \beta^{\ast \ast} \rangle \text{dim } \rho_e \text{ (by the orthogonality relations,)} = \langle \gamma \otimes \delta^* \gamma', \gamma' \otimes \beta^{\ast \ast} \rangle \text{ in } C_\rho. \text{ In } C_\rho, \langle \gamma \otimes \delta^* \rangle \circ (\gamma' \otimes \beta^{\ast \ast})(\alpha) = \langle \alpha, \beta \rangle \langle \gamma', \delta \rangle \gamma. \text{ To show } T_\rho \text{ is a homomorphism we need } T_{\rho\gamma} \ast T_{\rho\gamma'} = \langle \gamma', \delta \rangle T_{\rho\gamma'} \text{dim } \rho_e. \text{ We compute}
\]
\[
T_{\rho\gamma} \ast T_{\rho\gamma'}(\Phi_{q'}) = \int \int_{\varepsilon_{qr}} \langle \gamma_q, \rho(q) \delta_q \rangle \frac{d\Phi_q}{\dim \rho_e} \langle \gamma_{q'}, \rho(q')^{-1} \Phi_{q'} \delta_{q'} \rangle d\Phi_{q'} \\
= \int \langle \gamma_q, \rho(\Phi_q) \delta_q \rangle \frac{d\Phi_q}{\dim \rho_e} \langle \gamma_{q'}, \delta_{q'} \rangle d\Phi_{q'} \text{dim } \rho_e = T_{\rho\gamma'}(\langle \gamma', \gamma \rangle) \text{dim } \rho_e
\]
as desired. Finally, it is easy to show that
\[
T_\rho(\gamma \otimes \delta^{\ast \ast}) = (T_\rho(\gamma \otimes \delta^{\ast \ast}))^*.\]

**Theorem 6.5.** Let \( \varepsilon \) be a set of irreducible representations of \( Z \) containing exactly one member from each equivalence class. Then \( \sum_{\rho \in \varepsilon} T_\rho \) is a \(*\) isomorphism and isometry of \( \sum_{\rho \in \varepsilon} C_\rho \) onto \( \mathcal{L}(Z) \).
Proof. The main point is that the functions $T_{pr\delta}$ for $\rho \in \mathcal{E}$, $\gamma$ and $\delta \in \Gamma_\varepsilon(E^\rho)$, separate the points of $Z$, and $T_{pr\delta}$ is orthogonal to $T_{pr\gamma}$ if $\rho \neq \rho'$ and $\rho$ and $\rho' \in \mathcal{E}$.

7. Remarks. 7.0. The algebra $C_*(Z)$ forms a quasi-unitary algebra as defined by Dixmier in [2] if we use the inner product

$$\langle f \cdot g \rangle = \int \int \int \sqrt{J(\Phi_{qp})} f(\Psi_{qr}) g(\Psi_{qr}) d\Psi_{qr} d\Phi_{qp}$$

and $f^* = f^*$, and

$$f^*(\Phi_{qp}) = f(\Phi_{qp})/\sqrt{J(\Phi_{qp})}.$$  

Then $C_*(Z)$ is essentially the same as the algebra Dixmier defines on page 310, [2] in the special case that $Z$ is the example of (3.5c). Also, in this special case, the representation defined in (4.4) is substantially the same as that defined by Glimm in Theorem 1.5, [4].

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References


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Charles A. Akemann, *Invariant subspaces of C(G)* ........................................ 421
Dan Amir and Zvi Ziegler, *Generalized convexity cones and their duals* .... 425
Raymond Balbes, *On (J, M, m)-extensions of order sums of distributive lattices* .......................................................... 441
Jan-Erik Björk, *Extensions of the maximal ideal space of a function algebra* .......................................................... 453
Frank Castagna, *Sums of automorphisms of a primary abelian group* ...... 463
Theodore Seio Chihara, *On determinate Hamburger moment problems* .... 475
Zeev Ditzian, *Convolution transforms whose inversion function has complex roots in a wide angle* ................................. 485
Myron Goldstein, *On a paper of Rao* .......................................................... 497
Velmer B. Headley and Charles Andrew Swanson, *Oscillation criteria for elliptic equations* ................................................. 501
John Willard Heidel, *Qualitative behavior of solutions of a third order nonlinear differential equation* ................................ 507
Alan Carleton Hindmarsh, *Pick’s conditions and analyticity* .................. 527
Bruce Ansgar Jensen and Donald Wright Miller, *Commutative semigroups which are almost finite* .................................... 533
Lynn Clifford Kurtz and Don Harrell Tucker, *An extended form of the mean-ergodic theorem* ............................................ 539
S. P. Lloyd, *Feller boundary induced by a transition operator* .............. 547
Henry B. Mann, Josephine Mitchell and Lowell Schoenfeld, *A new proof of the maximum principle for doubly-harmonic functions* ........ 567
Robert Einsohn Mosher, *The product formula for the third obstruction* .... 573
Sam Bernard Nadler, Jr., *Sequences of contractions and fixed points* ..... 579
Eric Albert Nordgren, *Invariant subspaces of a direct sum of weighted shifts* .......................................................... 587
Fred Richman, *Thin abelian p-groups* ...................................................... 599
Jordan Tobias Rosenbaum, *Simultaneous interpolation in H2. II* ........ 607
Charles Thomas Scarborough, *Minimal Urysohn spaces* ...................... 611
Malcolm Jay Sherman, *Disjoint invariant subspaces* ............................... 619
Joel John Westman, *Harmonic analysis on groupoids* ............................. 621
William Jennings Wickless, *Quasi-isomorphism and TFM rings* .......... 633
Minoru Hasegawa, *Correction to “On the convergence of resolvents of operators”* .......................................................... 641