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HARMONIC ANALYSIS ON GROUPOIDS

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This paper generalizes harmonic analysis on groups to obtain a theory of harmonic analysis on groupoids. A system of measures is obtained for a locally compact locally trivial groupoid, Z , analogous to left Haar measure for a locally compact group. Then a convolution and involution are defined on $C_c(Z)$ = the continuous complex valued functions on Z with compact support. Strongly continuous unitary representations of Z on certain fiber bundles, called representation bundles, are lifted to $C_c(Z)$, yielding $*$ representations of $C_c(Z)$. A norm, $\| \cdot \|_{12}$, is defined on $C_c(Z)$, and the convolution, involution, and representations all extend to $\mathcal{L}_{12}(Z)$ = the $\| \cdot \|_{12}$ completion of $C_c(Z)$. The main example given is that of the groupoid $Z = Z(G, H)$ that arises naturally from a Lie group G and a closed subgroup H . In this example, the representations of Z are related to induced representations of G . Finally, if Z_{ee} (=the group of elements in Z with left unit=right unit = e) is compact then we canonically represent $\mathcal{L}_2(Z)$ as a direct sum of certain simple H^* -algebras.

We use extensively the notation and results of [8], except that [8] assumes a C^r manifold structure on the groupoid Z , and we want to consider groupoids with just topological structure. There is no essential difficulty in developing the main results of [8] for locally trivial topological groupoids. In particular, a C^r coordinate (resp. C^r fiber) bundle in [8] becomes a coordinate (resp. fiber) bundle as defined in [7].

Reviewing [8, § 1], the algebraic structure of a (transitive) groupoid, Z (over M), consists of a subset M of Z (called the units of Z), a projection $l \times r$ of Z onto $M \times M$ sending $\Phi_{qp} \in Z$ into (left unit Φ_{qp} , right unit of Φ_{qp}) = (q, p) , and a law of composition defined for pairs Φ_{qp}, Ψ_{rs} such that $p = r$. For $B \subseteq M \times M$, Z_B is defined as $(l \times r)^{-1}(B)$, and $Z_{qp} = (l \times r)^{-1}(q, p)$. The composition $\Phi_{qp} \cdot \Psi_{ps} \in Z_{qs}$, and $(\Phi_{qp} \cdot \Psi_{ps}) \cdot \Gamma_{st} = \Phi_{qp} \cdot (\Psi_{ps} \cdot \Gamma_{st})$. The unit $q \in M$ may be written 1_{qq} , and $1_{qq} \cdot \Phi_{qp} = \Phi_{qp} \cdot 1_{pp} = \Phi_{qp}$. Also, Φ_{qp} has an inverse, Φ_{qp}^{-1} , such that $\Phi_{qp}^{-1} \cdot \Phi_{qp} = 1_{pp}$ and $\Phi_{qp} \cdot \Phi_{qp}^{-1} = 1_{qq}$.

A coordinate groupoid (Z, Σ_e) over M consists of the following:

- (1.1) An (algebraic, transitive) groupoid Z over M and a Hausdorff topological structure for M .
- (1.2) A distinguished point $e \in M$ and a Hausdorff topological group structure for the group Z_{ee} .
- (1.3) A set of functions $\Sigma_e = \{\alpha: U_\alpha \rightarrow Z_{U_\alpha \times e}\}$ such that U_α is

open in M and $l \cdot \alpha = \text{identity map}$, satisfying

$$(1.3.1) \quad \bigcup_{\alpha \in \Sigma_\alpha} U_\alpha = M.$$

(1.3.2) For α and $\beta \in \Sigma_e$, the map $g_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow Z_{ee}$; $g_{\alpha\beta}(q) = \alpha(q)^{-1} \circ \beta(q)$, is continuous.

Then the constructions of [8] lead to a topological structure for Z , making Z a locally trivial topological groupoid as defined by Ehresmann in [3]. Conversely, any such groupoid arises from a coordinate groupoid.

Finally, we stipulate that the letter “ Z ” will always represent a locally compact locally trivial groupoid. Note Z is locally compact if and only if both Z_{ee} and M are locally compact.

2. We first consider systems of measures on a groupoid, Z over M .

DEFINITION 2.1. A (continuous) *system of measures* on Z is an indexed set $\lambda = \{\lambda_{qp}: (q, p) \in M \times M\}$, where λ_{qp} is a regular Borel measure on Z_{qp} . We will write $\lambda_{qp}(f) = \int_Z f(\Phi_{qp}) d\lambda_{qp}$, where f is an integrable function on Z_{qp} , and will require that the function $\lambda(h): M \times M \rightarrow C$; $\lambda(h)(q, p) = \lambda_{qp}(h|_{Z_{qp}})$ be in $C_c(M \times M)$ whenever $h \in C_c(Z)$.

The concepts of “left and right invariance” are easily applied to systems of measures.

DEFINITION 2.2. A system of measures, λ , is said to be *left invariant* if and only if

$$(2.2.1) \quad \int_{Z_{rp}} f(\Psi_{qr} \cdot \Phi_{rp}) d\lambda_{rp} = \int_{Z_{qp}} f(\Gamma_{qp}) d\lambda_{qp},$$

for all $\Psi_{qr} \in Z$ and $p \in M$ and $f \in C_c(Z_{qp})$. Similarly, for *right invariance* the condition is (with $f \in C_c(Z_{pr})$):

$$(2.2.2) \quad \int_{Z_{pq}} f(\Phi_{pq} \cdot \Psi_{qr}) d\lambda_{pq} = \int_{Z_{pr}} f(\Gamma_{pr}) d\lambda_{pr}.$$

If Z_{ee} is unimodular, it is easy to obtain a left and right invariant system of measures for Z from a Haar measure on Z_{ee} (use (2.6.1) with $\Delta \equiv 1$). In the general case, we extend the modular function for Z_{ee} to Z , and then obtain a left invariant system of measures for Z (depending on the extension).

DEFINITION 2.3. A function $\Delta: Z \rightarrow R^+$ is called a *modular function* for Z if and only if:

(2.3.1) Δ is a continuous homomorphism (multiplicative structure for $R^+ = \text{real numbers } > 0$.)

(2.3.2) $\Delta|_{Z_{ee}}$ is the modular function for Z_{ee} .

THEOERM 2.4. *If M is paracompact, then there exists a modular function for Z . Given two modular functions, Δ and Δ' , on Z , we have $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{qp})$, $h: M \times M \rightarrow R^+$ is a continuous homomorphism (with the trivial groupoid structure on $M \times M$, see (3.5b)).*

Proof. Let Σ_e be a set of local sections in $Z_{M \times e}$ such that $\{U_\alpha = \text{dom } \alpha: \alpha \in \Sigma_e\}$ is a locally finite cover of M (using the paracompactness of M) and let $\{f_\alpha\}$ be a partition of 1 such that $\text{support}(f_\alpha) \subseteq U_\alpha \cdot \Delta_{ee}$ is the modular function for Z_{ee} . We define $\Delta = e^\delta$, where

$$\delta(\Phi_{qp}) = \sum_{f_\alpha, f_\beta} f_\alpha(q)f_\beta(p) \log (\Delta_{ee}(\alpha(q)^{-1} \cdot \Phi_{qp} \cdot \beta(p))) .$$

Then Δ is a modular function for Z . Given a continuous homomorphism $h: M \times M \rightarrow R^+$, then Δ' defined by $\Delta'(\Phi_{qp}) = h(q, p)\Delta(\Phi_{qp})$ is a modular function for Z . Conversely, given two modular functions Δ and Δ' on Z , we find that $h(q, p) = \Delta'(\Phi_{qp})/\Delta(\Phi_{qp})$ is independent of Φ_{qp} for the given units, and that $h: M \times M \rightarrow R^+$ is a continuous homomorphism.

THEOREM 2.5. *If λ is a left (resp. right) invariant system of measures on Z , then λ_{qq} is a left (resp. right) Haar measure on Z_{qq} for each $q \in M$.*

From here on we assume λ_{ee} is a *fixed* left Haar measure on Z_{ee} , and will write $\lambda_{ee}(f) = \int_{Z_{ee}} f(\Phi_{ee})d\Phi_{ee}$.

THEOREM 2.6. *There is a natural one-to-one correspondence between the left invariant systems of measures on Z and the modular functions on Z .*

Proof. Given a modular function, Δ , on Z , we define the system of measures, λ , by

$$\begin{aligned} \lambda_{qp}(f) &= \int_{Z_{qp}} f(\Phi_{qp})d\Phi_{qp} \\ (2.6.1) \qquad &= \int_{Z_{ee}} \Delta(\Gamma_{ep})f(\Psi_{qe}\Delta_{ee}\Gamma_{ep})d\Delta_{ee} . \end{aligned}$$

λ_{q_p} is independent of the choice of Ψ_{q_e} and Γ_{e_p} with the indicated units, and λ is left invariant. Conversely if λ is a left invariant system of measures the above equation defines Δ on $Z_{e \times M}$. Then Δ may be extended to a continuous homomorphism of Z into R^+ , and $\Delta|_{Z_{ee}}$ is the modular function of Z_{ee} .

THEOREM 2.7. *If Z_{ee} is unimodular, then there is a unique left and right system of measures on Z (recall λ_{ee} is a fixed left Haar measure).*

Proof. Just choose $\Delta \equiv 1$.

From here on we will assume that a fixed modular function Δ has been given for Z , and the corresponding left invariant system of measures is λ as defined in (2.6.1). A fixed regular Borel measure, μ , is specified for M , and $\mu(f)$ will be written $\int_M f(q)dq$, for any integrable function f on M . We require support of $\mu = M$.

3. DEFINITION 3.1. Given f and $g \in C_c(Z)$ we define the *convolution* of f and g , $f * g$, by $f * g(\Phi_{q_p}) = \int_M \int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}^{-1} \cdot \Phi_{q_p})d\Psi_{qr}dr$.

THEOREM 3.2. *$C_c(Z)$ forms an algebra over C with convolution as the law of multiplication, and the usual addition and scalar multiplication.*

Proof. The main points to verify are:

- (a) $f * g \in C_c(Z)$ and
- (b) $(f * g) * h = f * (g * h)$.

In regard to (a), if support $(f) \subseteq A$ and support $(g) \subseteq B$ then it is easy to show that support $(f * g) \subseteq A \cdot B$. $A \cdot B$ is the image of $(A \times B) \cap D \subseteq Z \times Z$ under composition, where D is the (closed) subset of $Z \times Z$ where composition is defined. Hence $A \cdot B$ is compact if A and B are compact.

In regard to (b), we compute $(f * g) * h(\Phi_{q_p})$

$$= \int_M \int_{Z_{qs}} \left(\int_M \int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}^{-1} \cdot \Gamma_{qs})d\Psi_{qr}dr \right) h(\Gamma_{qs}^{-1} \cdot \Phi_{q_p})d\Gamma_{qs}ds .$$

Substitute $A_{rs} = \Psi_{qr}^{-1} \cdot \Gamma_{qs}$, and interchange the order of integration to obtain

$$\begin{aligned} &= \int_M \int_{Z_{qr}} f(\Psi_{qr}) \left(\int_M \int_{Z_{rs}} g(A_{rs})h(A_{rs}^{-1} \cdot \Psi_{qr}^{-1} \cdot \Phi_{q_p})dA_{rs}ds \right) d\Psi_{qr}dr \\ &= f * (g * h)(\Phi_{q_p}) . \end{aligned}$$

Next, we define an involution for $C_c(Z)$.

DEFINITION 3.3. Given $f \in C_c(Z)$, we define f^* by

$$f^*(\Phi_{qp}) = \bar{f}(\Phi_{qp}^{-1})\Delta(\Phi_{qp}^{-1})$$

(where \bar{f} is the complex conjugate of f).

THEOREM 3.4. The map $f \rightarrow f^*: C_c(Z) \rightarrow C_c(Z)$ is an involution (see [6]).

Proof. The only difficult part is to show $(f * g)^* = g^* * f^*$. We compute

$$\begin{aligned} (f * g)^*(\Phi_{qp}) &= \int_M \int_{Z_{pr}} \bar{f}(\Psi_{pr}) \bar{g}(\Psi_{pr}^{-1} \cdot \Phi_{qp}^{-1}) \Delta(\Phi_{qp}^{-1}) d\Psi_{pr} dr \\ &= (\text{substituting } \Gamma_{qr} = \Phi_{qp} \cdot \Psi_{pr}) \\ &\int_M \int_{Z_{qr}} \bar{g}(\Gamma_{qr}^{-1}) \bar{f}(\Phi_{qp}^{-1} \cdot \Gamma_{qr}) \Delta(\Phi_{qp}^{-1}) d\Gamma_{qr} dr = (g^* * f^*)(\Phi_{qp}). \end{aligned}$$

EXAMPLES 3.5. (a) Suppose $M = \{e\}$ and $\mu(1) = 1$. Then $Z = Z_{ee}$ is a locally compact group, $f * g$ is the ordinary convolution, and $f \rightarrow f^*$ is the usual involution.

(b) Suppose $Z = M' \times M'$ and $M = \text{diagonal of } M' \times M'$. We define the *trivial groupoid* structure for Z over M as follows:

$$l(q, p) = (q, q) \quad \text{and} \quad r(q, p) = (p, p),$$

composition is given by $(q, p) \cdot (p, r) = (q, r)$, and $(q, q) \rightarrow (q, e)$ gives a global section of $l: Z_{M \times e} \rightarrow M$.

If M' is discrete, then f and $g \in C_c(Z)$ are matrices indexed by M' , with a finite number of nonzero entries. If $\mu(\{q\}) = 1$ for all $q \in M$, and $\lambda_{ee}(1) = 1$, then $f * g$ is the matrix composition of f and g .

(c) Suppose G is a Lie group and H is a closed subgroup of G . We define the *homogeneous space groupoid* for G and H , $Z(G, H) = Z = \{(q, \Phi, p): \Phi \in G, p \text{ and } q \in G/H, \text{ and } \Phi p = q\}$. The groupoid structure for Z is given as follows: $M = \{(q, 1, q): q \in G/H\}$ is the set of units, and $q \rightarrow (q, 1, q)$ identifies M with G/H to give M the required topology; $l(q, \Phi, p) = (q, 1, q)$ and $r(q, \Phi, p) = (p, 1, p)$. Composition is defined by $(q, \Phi, p) \cdot (p, \Psi, r) = (q, \Phi \cdot \Psi, r)$; the local sections of $l: Z_{M \times e} \rightarrow M$ come from local sections of $G \rightarrow G/H$ (identifying G/H with M as above, and taking $e = 1H$): $(e, \Phi, e) \rightarrow \Phi$ is a group isomorphism sending Z_{ee} onto H , giving Z_{ee} the required topology.

We note that $Z_{M \times e}$ is essentially the usual principal bundle obtained from G and H .

For simplicity we only consider in this paper the case where Δ_H (the modular function for H) = Δ_G (the modular function for G), restricted to H . Then, by a theorem in [5, Chapter 10], there is a G

invariant measure on M , which we take for μ . There is a canonical (continuous) homomorphism $\zeta: Z \rightarrow G$, defined by $\zeta(q, \Phi, p) = \Phi$. Note that ζ maps Z onto G , and that $\zeta|_{Z_{ee}}$ is an isomorphism mapping Z_{ee} onto H . The above consideration leads to the following:

THEOREM 3.5.1. $\Delta_G \cdot \zeta$ is a modular function for Z . Unless otherwise mentioned we will always use $\Delta = \Delta_G \cdot \zeta$ for $Z(G, H)$.

If M is compact and $\mu(1) = 1$, then $\zeta^*(f) = f \cdot \zeta \in C_c(Z)$ for $f \in C_c(Z)$, and we obtain the

THEOREM 3.5.2. $\zeta^*: C_c(G) \rightarrow C_c(Z)$ is a one-to-one* homomorphism (with the usual convolution and involution on $C_c(G)$, using a suitable left Haar measure on G).

Proof. The first point is that $f \rightarrow \int_M \int_{Z_{qp}} \zeta^*(f)(\Phi_{qp}) d\Phi_{qp} dp$ (writing $(q, \Phi, p) = \Phi_{qp}$) defines a left invariant measure on G which we take as the desired left Haar measure on G . Note, this measure on G is independent of the choice of $q \in M$. Next, we compute

$$\begin{aligned} \zeta^*(f) * \zeta^*(g)(\Phi_{qp}) &= \int_{Z_q \times M} \zeta^*(f)(\Psi_{qr}) \zeta^*(g)(\Psi_{qr}^{-1} \cdot \Phi_{qp}) d\Psi_{qr} dr \\ &= \int_G f(\Psi) g(\Psi^{-1} \cdot \Phi) d\Psi \\ &= (f * g)(\Phi) = \zeta^*(f * g)(\Phi_{qp}), \text{ as required.} \end{aligned}$$

Finally, for $f \in C_c(G)$,

$$(\zeta^*(f))^*(\Phi_{qp}) = (\zeta^*(f))(\Phi_{qp}^{-1}) \Delta(\Phi_{qp}^{-1}) = f(\Phi^{-1}) \Delta_G(\Phi^{-1}) = \zeta^*(f^*)(\Phi_{qp}),$$

as required.

4. DEFINITION 4.1. A (unitary) *representation bundle*, E , is a fiber bundle with a Hilbert space structure for the fiber Y , and group $U(Y)$ = the unitary operators on Y with the strong operator topology.

We note that there is a natural *inner product field*, \langle, \rangle , on E . For $q \in M$, \langle, \rangle_q is an inner product on E_q defined via any admissible map from the fiber Y . Then \langle, \rangle_q makes E_q a Hilbert space and the unitary maps from Y to E_q are the admissible maps from Y to E_q .

Using the given regular Borel measure, μ , on M , we obtain an inner product on $\Gamma_c(E)$, the continuous sections in E with compact support. For γ and $\delta \in \Gamma_c(E)$,

$$\langle \gamma, \delta \rangle = \int_M \langle \gamma_q, \delta_q \rangle_q dq .$$

The completion of $\Gamma_c(E)$ with respect to this inner product is then a Hilbert space, to be called $\Gamma_2(E)$.

DEFINITION 4.2. A (strongly continuous) *unitary representation* ρ of Z on a representation bundle E is a continuous homomorphism $\rho: Z \rightarrow A(E) =$ the (locally trivial) groupoid of admissible maps between the fibers of E , such that ρ is the identity map on the units of Z (see [8]).

The main results listed below are obtained essentially as in [8, § 4].

THEOREMS 4.3. (a) *If ρ is given as in (4.2) then $\rho|_{z_{ee}} = \rho_e$ defines a unitary representation of Z_{ee} on E_e .*

(b) *Given a unitary representation ρ_e of Z_{ee} on a Hilbert space E_e , there is a representation bundle E' and representation ρ' of Z on E' such that $\rho'|_{z_{ee}} \cong \rho_e$ (a unitary equivalence).*

(c) *Two representations ρ and ρ' of Z on E and E' respectively are equivalent (as in [8]) if and only if $\rho|_{z_{ee}} \cong \rho'|_{z_{ee}}$.*

A groupoid representation, ρ , of Z on E^ρ defines a representation of the algebra $C_c(Z)$; $\rho: C_c(Z) \rightarrow \mathcal{L}(\Gamma_2(E^\rho)) =$ the bounded linear maps of $\Gamma_2(E^\rho)$ into itself.

DEFINITION 4.4. Given $f \in C_c(Z)$ and $\gamma \in \Gamma_c(E^\rho)$, we define $\rho(f)\gamma$ by $(\rho(f)\gamma)_q = \int_M \int_{Z_{qp}} f(\Phi_{qp}) \rho(\Phi_{qp}) \gamma_p d\Phi_{qp} dp$. Alternatively,

$$\langle \rho(f)\gamma, \delta \rangle = \int_M \int_M \int_{Z_{qp}} f(\Phi_{qp}) \langle \rho(\Phi_{qp}) \gamma_p, \delta_q \rangle d\Phi_{qp} dq dp .$$

THEOREM 4.5. $\|\rho(f)\gamma\|_2 \leq \|f\|_{12} \|\gamma\|_2$, where

$$\|f\|_{12}^2 = \int_{M \times M} \int_{Z_{qp}} |f(\Phi_{qp})|^2 d\Phi_{qp} .$$

Proof. See (5.4). Accordingly $\rho(f)$ extends to a bounded operator on $\Gamma_2(E^\rho)$ of norm $\leq \|f\|_{12}$. $\mathcal{L}(\Gamma_2(E^\rho))$ has a natural Banach* algebra structure.

THEOREM 4.6. *The representation $\rho: C_c(Z) \rightarrow \mathcal{L}(\Gamma_2(E^\rho))$ is a *-homomorphism.*

Proof. For f and $g \in C_c(Z)$, we compute

$$\begin{aligned}
 (\rho(f * g)\gamma)_q &= \int_M \int_{Z_{qp}} \left(\int_M \int_{Z_{qr}} f(\Psi_{qr}) g(\Psi_{qr}^{-1} \cdot \Phi_{qp}) d\Psi_{qr} dr \right) \rho(\Phi_{qp}) \gamma_p d\Phi_{qp} dp \\
 &= (\text{substituting } \Gamma_{rp} = \Psi_{qr}^{-1} \cdot \Phi_{qp} \text{ and interchanging the} \\
 &\quad \text{order of integration}) \\
 &\int_M \int_{Z_{qr}} f(\Psi_{qr}) \rho(\Psi_{qr}) \left(\int_M \int_{Z_{rp}} g(\Gamma_{rp}) \rho(\Gamma_{rp}) \gamma_p d\Gamma_{rp} dp \right) d\Psi_{qr} dr \\
 &= (\rho(f)(\rho(g)\gamma))_q \text{ as desired.}
 \end{aligned}$$

Finally, we compute

$$\begin{aligned}
 \langle \rho(f^*)\gamma, \delta \rangle &= \int_{M \times M} \int_{Z_{qp}} f^*(\Phi_{qp}) \langle \rho(\Phi_{qp})\gamma_p, \delta_q \rangle_q d\Phi_{qp} dp dq \\
 &= \int_{M \times M} \int_{Z_{qp}} \bar{f}(\Phi_{qp}^{-1}) \Delta(\Phi_{qp}^{-1}) \langle \rho(\Phi_{qp}^{-1})\delta_q, \gamma_p \rangle_p^- d\Phi_{qp} dp dq \\
 (\text{see (5.2.1)}) &= \int_{M \times M} \int_{Z_{pq}} \bar{f}(\Psi_{pq}) \langle \rho(\Psi_{pq})\delta_q, \gamma_p \rangle_p^- d\Psi_{pq} dp \\
 &= \langle \gamma, \rho(f)\delta \rangle, \text{ so } \rho(f^*) = \rho(f)^* .
 \end{aligned}$$

The following example provides a representation analogous to the left regular representation for groups.

EXAMPLE 4.7. Let ρ_e be the strongly continuous unitary representation of Z_{ee} on $\mathcal{L}_2(Z_{e \times M})$ given by $(\rho_e(\Phi_{ee})f_e)(\Psi_{ep}) = f_e(\Phi_{ee}^{-1} \cdot \Psi_{ep})$. The representation bundle F arising from ρ_e and Z may be regarded as $\cup_{q \in M} \mathcal{L}_2(Z_{q \times M})$. The map $f \rightarrow f'; C_c(Z) \rightarrow \Gamma_c(F)$, defined by $f'(q) = f|_{q \times M}$ is bijective, and $\|f\|_2 = \|f'\|_2$. Accordingly, we can identify $\mathcal{L}_2(Z)$ and $\Gamma_2(F)$. Given f and $g \in C_c(Z)$, then $\rho(f)g' = (f * g)'$.

5. **DEFINITION 5.1.** For $f \in C_c(Z)$, we define

$$\|f\|_{12} = \left(\int_M \int_M \left(\int_{Z_{qp}} |f(\Phi_{qp})| d\Phi_{qp} \right)^2 dq dp \right)^{\frac{1}{2}} .$$

$\| \cdot \|_{12}$ defines a norm on $C_c(Z)$; we complete $C_c(Z)$ with respect to $\| \cdot \|_{12}$ to form $\mathcal{L}_{12}(Z)$.

To simplify matters, we recall the map: $\lambda: C_c(Z) \rightarrow C_c(M \times M)$, where $\lambda(f)(q, p) = \int_{Z_{qp}} f(\Phi_{qp}) d\Phi_{qp}$.

THEOREM 5.2. $\lambda(f * g) = \lambda(f) * \lambda(g)$ and $\lambda(f^*) = \lambda(f)^*$, using the trivial groupoid structure on $M \times M$ over the diagonal of $M \times M$. (on $(M \times M)_{ee} = \{(e, e)\}$ the Haar measure is taken as 1).

Proof. We write f_{qp} for $\lambda(f)(q, p)$. Then

$$\begin{aligned} \lambda(f * g)(q, p) &= \int_{Z_{qp}} \int_M \int_{Z_{qr}} f(\Psi_{qr})g(\Psi_{qr}^{-1} \cdot \Phi_{qp})d\Psi_{qr}drd\Phi_{qp} \\ &= \int_M \int_{Z_{qr}} f(\Psi_{qr})g_{rp}d\Psi_{qr}dr \\ &= \int_M f_{qr}g_{rp}dr = (\lambda(f) * \lambda(g))(\Phi_{qp}) . \end{aligned}$$

Next, to show $\lambda(f^*) = \lambda(f)^*$ we should show

$$(5.2.1) \quad \int_{Z_{qp}} f(\Phi_{qp}^{-1})\Delta(\Phi_{qp}^{-1})d\Phi_{qp} = \int_{Z_{pq}} f(\Phi_{pq})d\Phi_{pq} .$$

If $p = q = e$ this is a standard theorem. The extension to the general case is routine, using (2.6.1).

Accordingly, $f \rightarrow \lambda(f)$ defines a $*$ homomorphism. Also, $\|f\|_{12} = \|\lambda(|f|)\|_2$, where $\|\cdot\|_2$ is the \mathcal{L}_2 norm on $C_c(M \times M)$. For f and $g \in C_c(M \times M)$ it is easy to show that $\|f * g\|_2 \leq \|f\|_2 \|g\|_2$. Finally, we obtain the

THEOREM 5.3. *Given f and $g \in C_c(Z)$ then $\|f * g\|_{12} \leq \|f\|_{12} \|g\|_{12}$ and $\|f\| = \|f^*\|$.*

Proof.

$$\|\lambda(|f * g|)\|_2 \leq \|\lambda(|f| * |g|)\|_2 = \|\lambda(|f|) * \lambda(|g|)\|_2 \leq \|f\|_{12} \|g\|_{12}$$

settles the first part, and $\|\lambda(|f^*|)\|_2 = \|\lambda(|f|)^*\|_2 = \|\lambda(|f|)\|_2$ settles the second part.

Accordingly, the convolution and $(*)$ involution extend to $\mathcal{L}_{12}(Z)$, making $\mathcal{L}_{12}(Z)$ a Banach algebra with a natural involution. Representations also extend to $\mathcal{L}_{12}(Z)$ as shown below.

THEOREM 5.4. *For $f \in C_c(Z)$ and $\gamma \in \Gamma_c(E)$, $\|\rho(f)\gamma\|_2 \leq \|f\|_{12} \|\gamma\|_2$.*

Proof. $\langle \rho(f)\gamma, \rho(f)\gamma \rangle$

$$\begin{aligned} &= \int_M \int_M \int_M \int_{Z_{qr}} \int_{Z_{qp}} f(\Phi_{qp})\bar{f}(\Psi_{qr})\langle \rho(\Phi_{qp})\gamma_p, \rho(\Psi_{qr})\gamma_r \rangle d\Psi_{qr}d\Psi_{qp}drdpdq \\ &\leq \int_{M \times M \times M} |f_{qp}| \|\gamma_p\| |f_{qr}| \|\gamma_r\| drdpdq \\ &= \int_M \left(\int_M |f_{qp}| \|\gamma_p\| dp \right) \left(\int_M |f_{qr}| \|\gamma_r\| dr \right) dq \\ &\leq \int_M \left(\int_M |f_{qp}| \|\gamma_p\| dp \right)^2 dq \end{aligned}$$

$$\begin{aligned} &\leq \int_M \left(\int_M |f_{qp}|^2 dp \int_M \|\gamma_p\|^2 dp \right) dq \\ &= \|f\|_{12}^2 \|\gamma\|_2^2. \end{aligned}$$

Accordingly, ρ of Z on E lifts to a $*$ representation of $\mathcal{L}_2(Z)$ on $\Gamma_2(E)$.

EXAMPLE 5.5. Suppose $Z = Z(G, H)$ as in (3.5 c), and that G/H is compact and $\mu(1) = 1$. Then $\zeta^*: C_c(G) \rightarrow C_c(Z)$ (see (3.5.2) is a norm increasing $*$ homomorphism.

Furthermore, a representation ρ of Z on E defines a representation ρ' of G on $\Gamma_2(E)$, by $(\rho'(\Phi)\gamma)_q = \rho(\Phi_{qp})\gamma_p$, where $p = \Phi^{-1}(q)$ and $\Phi_{qp} = (q, \Phi, p)$. ρ' is a unitary representation since μ is invariant under G . Then ρ' is the induced representation (well known in group theory) from the representation ρ_e of $Z_{ee}(\cong H)$ on E_e . The diagram below, relating Z and G , commutes.

$$\begin{array}{ccc} C_c(Z) & \xrightarrow{\rho} & \mathcal{L}(\Gamma_2(E)) \\ \zeta^* \uparrow & & \parallel \\ C_c(G) & \xrightarrow{\rho'} & \mathcal{L}(\Gamma_2(E)). \end{array}$$

Note that the case $H = G, \mu(1) = \lambda_{ee}(1) = 1$, is the same as the Example 3.5a, where $Z = Z_{ee}$.

6. Suppose Z_{ee} is compact, $\Delta \equiv 1$, and $\lambda_{ee}(1) = 1$ (the vertically compact case). Then the completion of $C_c(Z)$ with respect to the $\|\cdot\|_2$ norm forms the Hilbert space $\mathcal{L}_2(Z)$. We will extend the “orthogonality relations” for compact groups to the above case, and represent $\mathcal{L}_2(Z)$ as a direct sum of simple H^* algebras.

DEFINITION 6.1. Given γ and $\delta \in \Gamma_c(E^\circ)$, where ρ is a representation of Z on E° , we define $T_{\rho\gamma\delta}: Z \rightarrow C$, by

$$T_{\rho\gamma\delta}(\Phi_{qp}) = \langle \gamma_q, \rho(\Phi_{qp})\delta_p \rangle_q.$$

THEOREM 6.2. *If ρ_e and ρ'_e are irreducible, then*

$$\langle T_{\rho\gamma\delta}, T_{\rho'\gamma'\delta'} \rangle = \begin{cases} \frac{\langle \gamma, \gamma' \rangle \langle \delta', \delta \rangle}{\dim \rho_e} & \text{if } \rho = \rho' \\ 0 & \text{if } \rho \text{ is not equivalent to } \rho'. \end{cases}$$

Proof. Integrating both sides of (6.2.1) over $M \times M$ yields the desired result.

$$\begin{aligned}
 (6.2.1) \quad & \int_{Z_{qp}} \langle \gamma_q, \rho(\Phi_q)\delta_p \rangle_q \langle \gamma'_q, \rho'(\Phi_{qp})\delta'_p \rangle_{q'} d\Phi_{qp} \\
 & = \begin{cases} \frac{\langle \gamma_q, \gamma'_q \rangle \langle \delta'_p, \delta_p \rangle}{\dim \rho_e} & \text{if } \rho = \rho' \\ 0 & \text{if } \rho \text{ is not equivalent to } \rho'. \end{cases}
 \end{aligned}$$

For $q = p = e$, (6.2.1) is just the orthogonality relations for compact groups. The proof of (6.2.1) for general p and q is similar to the usual derivation of the orthogonality relations, for example see [1].

Notation. The representations ρ and ρ' of Z on E^ρ and $E^{\rho'}$ respectively will be such that ρ_e and ρ'_e are irreducible. The map $\delta \rightarrow \delta^*$: $\Gamma_2(E) \rightarrow \Gamma_2(E)^* = \text{dual of } \Gamma_2(E)$, is defined by $\delta^*(\gamma) = \langle \gamma, \delta \rangle$. $\Gamma_c(E)^*$ is the image of $\Gamma_c(E)$ under $\delta \rightarrow \delta^*$. The (algebraic) tensor product $\Gamma_c(E^\rho) \otimes \Gamma_c(E^{\rho'})^*$ may be regarded as a (dense) subalgebra of $C_\rho =$ the Schmidt operators on $\Gamma_2(E^\rho)$. In particular $(\gamma \otimes \delta^*)(\beta) = \langle \beta, \delta \rangle \gamma$. Conversely, α and $\beta \in C_\rho$ can be regarded as elements of the (Hilbert space) tensor product $\Gamma_2(E^\rho) \otimes \Gamma_2(E^{\rho'})^*$. The inner product on C_ρ is defined by $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle \dim \rho_e$ where \langle , \rangle is the inner product on $\Gamma_2(E^\rho) \otimes \Gamma_2(E^{\rho'})^*$, making C_ρ a simple H^* algebra.

THEOREM 6.4. *The canonical map $T_\rho: \Gamma_c(E^\rho) \otimes \Gamma_c(E^\rho) \rightarrow C_c(Z)$ defined by $T_\rho(\gamma \otimes \delta^*) = T_{\rho\gamma\delta} \dim \rho_e$ extends to a $*$ homomorphism and isometry of C_ρ into $\mathcal{L}_2(Z)$.*

Proof. To show T_ρ defines an isometry from C_ρ we compute $\langle T_{\rho\gamma\delta} \dim \rho_e, T_{\rho\gamma'\beta} \dim \rho_e \rangle = \langle \gamma \otimes \delta^*, \gamma' \otimes \beta^* \rangle \dim \rho_e$ (by the orthogonality relations,) $= \langle \gamma \otimes \delta^*, \gamma' \otimes \beta^* \rangle$ in C_ρ . In C_ρ , $(\gamma \otimes \delta^*) \circ (\gamma' \otimes \beta^*)(\alpha) = \langle \alpha, \beta \rangle \langle \gamma', \delta \rangle \gamma$. To show T_ρ is a homomorphism we need $T_{\rho\gamma\delta} * T_{\rho\gamma'\beta} = \langle \gamma', \delta \rangle T_{\rho\gamma\beta} / \dim \rho_e$. We compute

$$\begin{aligned}
 T_{\rho\gamma\delta} * T_{\rho\gamma'\beta}(\Phi_{qp}) & = \int_M \int_{Z_{qr}} \langle \gamma_q, \rho(\Psi_{qr})\delta_r \rangle \langle \gamma'_r, \rho(\Psi_{qr}^{-1} \cdot \Phi_{qp})\beta_p \rangle d\Psi_{qr} dr \\
 & = \int_M \langle \gamma_q, \rho(\Phi_{qp})\beta_p \rangle \langle \gamma'_r, \delta_r \rangle dr / \dim \rho_e = T_{\rho\gamma\beta}(\langle \gamma', \gamma \rangle / \dim \rho_e)
 \end{aligned}$$

as desired. Finally, it is easy to show that

$$T_\rho((\gamma \otimes \delta^*)^*) = (T_\rho(\gamma \otimes \delta^*))^*.$$

THEOREM 6.5. *Let \mathcal{E} be a set of irreducible representations of Z containing exactly one member from each equivalence class. Then $\sum_{\rho \in \mathcal{E}} T_\rho$ is a $*$ isomorphism and isometry of $\sum_{\rho \in \mathcal{E}} C_\rho$ onto $\mathcal{L}_2(Z)$.*

Proof. The main point is that the functions $T_{\rho\gamma\delta}$ for $\rho \in \mathcal{E}$, γ and $\delta \in \Gamma_c(E^\rho)$, separate the points of Z , and $T_{\rho\gamma\delta}$ is orthogonal to $T_{\rho'\gamma'\delta'}$ if $\rho \neq \rho'$ and ρ and $\rho' \in \mathcal{E}$.

7. REMARKS. 7.0. The algebra $C_c(Z)$ forms a quasi-unitary algebra as defined by Dixmier in [2] if we use the inner product

$$\langle f \cdot g \rangle = \int_M \int_M \int_{Z_{qr}} \sqrt{\Delta(\Phi_{qp})} f(\Psi_{qr}) \bar{g}(\Psi_{qr}) d\Psi_{qr} dq dr, \quad f^* = f^s, \quad \text{and} \\ f^j(\Phi_{qp}) = f(\Phi_{qp}) / \sqrt{\Delta(\Phi_{qp})}.$$

Then $C_c(Z)$ is essentially the same as the algebra Dixmier defines on page 310, [2] in the special case that Z is the example of (3.5c). Also, in this special case, the representation defined in (4.4) is substantially the same as that defined by Glimm in Theorem 1.5, [4].

The author is indebted to the referee for the above references ([2] and [4]).

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