ON THE DECOMPOSITION OF INFINITELY DIVISIBLE PROBABILITY LAWS WITHOUT NORMAL FACTOR

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In the theory of the decomposition of probability laws, the fundamental problem stated by D. A. Raikov of the characterization of the class \( I_\circ \) of the infinitely divisible laws without indecomposable factors has been studied in the case of univariate laws by Yu. V. Linnik and I. V. Ostrovskiy. Lately, we have shown that nearly all these results can be extended to the case of multivariate laws. In this paper, we give a result which can be considered as an extension of a theorem of Raikov and P. Lévy and of a particular case of theorems of Linnik, and the extension of this result to the case of several variables.

If we consider the finite products of Poisson laws, i.e., the characteristic functions of the variable \( t \) of the form

\[
f(t) = \exp \left\{ ict + \sum_{j=1}^{p} \lambda_j \left[ \exp (i\alpha_j t) - 1 \right] \right\}
\]

\((c \text{ real, } \lambda_j > 0, \alpha_j > 0)\), three general results are known, the first being owed to D. A. Raikov [9] and P. Lévy [4] and the third to Yu. V. Linnik [5, Chapter 9]:

(a) if \( \alpha_1, \ldots, \alpha_p \) are rationally independent, \( f \) has no indecomposable factor;
(b) if \( \alpha_1, \ldots, \alpha_p \) are such that \( 0 < a \leq \alpha_1 \leq 2a \) \((j = 1, \ldots, p)\), \( f \) has no indecomposable factor;
(c) if \( \alpha_{j+1}/\alpha_j \) is an integer greater than 1 \((j = 1, \ldots, p - 1)\), \( f \) has no indecomposable factor.

Lately, I. V. Ostrovskiy [8] has extended the two results (a) and (b) of Raikov and Lévy to the case of a continuous spectrum, the base of his study being the

THEOREM 1. (see also [1] chapter 8). Let \( f_0 \) be the infinitely divisible characteristic function of the variable \( t \) defined by

\[
f_0(t) = \exp \left\{ i\gamma t + \int_{a}^{b} \left[ \exp (i\gamma t) - 1 \right] d\mu(x) \right\},
\]

where \( \gamma \) is a real constant and \( \mu \) is a nonnegative measure defined on the segment \([a, b]\) \((0 < a < b < \infty)\). If \( f_1 \) is a factor of \( f_0 \), then

\[
f_1(t) = \exp \left\{ i\gamma t + \int_{a}^{b} \left[ \exp (i\gamma t) - 1 \right] d\mu(x) \right\},
\]
where \( c \) is a real constant and \( m \) is a measure defined on the segment \([a, b]\) which is nonnegative on \([a, 2a]\). Moreover,
\[
S(m) \subset [a, b] \cap (\infty S(\mu)) ,
\]
where \( S(N) \) means the support of a measure \( N \) and \((\infty A)\) is defined by
\[
(1)A = A ; \quad (p)A = (p - 1)A + A ; \quad (\infty A) = \bigcup_{p=1}^{\infty} (p)A
\]
(the symbol + indicates the vectorial sum of two subsets of \( \mathbb{R} \)).

He gives also a more general result which can be stated in the following manner:

**Theorem 2.** Let \( f_0 \) be the infinitely divisible characteristic function of the variable \( t \) defined by
\[
f_0(t) = \exp \left\{ i\gamma t + \int_a^b \left[ \exp (ixt) - 1 \right] d\mu(x) + \sum_{k=1}^{\infty} \lambda_k \left[ \exp (i\alpha_k t) - 1 \right] \right\} ,
\]
where \( \gamma \in \mathbb{R}, \lambda_k \geq 0, \alpha_k > 0 \) \((k = 1, 2, \ldots)\) and where the following conditions are satisfied:
1. the measure \( \mu \) is a nonnegative measure defined on the segment \([a, b]\) \((0 < a < b < \infty)\);
2. there exists a positive constant \( K \) such that
\[
\lambda_k = 0[\exp (-K \alpha_k^2)] \quad (k \to +\infty) ;
\]
3. \( \alpha_i > b \) and \( \alpha_{k+1}/\alpha_k \) is an integer greater than 1 \((k = 1, 2, \ldots)\).

If \( f_1 \) is a factor of \( f_0 \), then
\[
f_1(t) = \exp \left\{ict + \int_a^b \left[ \exp (ixt) - 1 \right] dm(x) + \sum_{k=1}^{\infty} l_k \left[ \exp (i\alpha_k t) - 1 \right] \right\} ,
\]
where \( c \) is a real constant and the following conditions are satisfied:
(a) \( 0 \leq l_k \leq \lambda_k \) \((k = 1, 2, \ldots)\);
(b) the measure \( m \) is a measure defined on the segment \([a, b]\) which is nonnegative on \([a, 2a]\) and such that
\[
m([b]) \geq 0 , \quad S(m) \subset [a, b] \cap (\infty S(\mu)) .
\]

Using the Theorems 1 and 2, we give in § 2, two theorems which can be considered as extensions of the results (a) and (c) stated above. Using the auxiliary results stated in § 3, we extend these results to the case of several variables in the § 4.

2. The case of one variable.

**Theorem 3.** Let \( f_0 \) be the infinitely divisible characteristic
function of the variable $t$ defined by

$$f_0(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{p} \sum_{k=1}^{r_j} \lambda_{j,k} [\exp (i\alpha_{j,k} t) - 1] \right\},$$

where $\gamma$ is a real constant, the $\lambda_{j,k}$ are nonnegative constants and the $\alpha_{j,k}$ are positive numbers satisfying the two conditions

(a) $\alpha_{j,k+1}/\alpha_{j,k}$ is an integer greater than 1 $(k = 1, \ldots, r_j - 1; j = 1, \ldots, p)$;

(b) $\alpha_{1,1}, \ldots, \alpha_{r,1}$ are rationally independent. If $f_1$ is a factor of $f_0$, then

$$f_1(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{p} \sum_{k=1}^{r_j} l_{j,k} [\exp (i\alpha_{j,k} t) - 1] \right\},$$

where $c$ is a real constant and the $l_{j,k}$ satisfy

$$0 \leq l_{j,k} \leq \lambda_{j,k}.$$

Proof. Let $f_1$ and $f_2$ be the two characteristic functions such that for any real $t$

(2.1) $$f_0(t) = f_1(t)f_2(t).$$

Since $f_0$ is an entire characteristic function, from Raikov's theorem ([6], theorem 8.1.1), $f_j$ $(j = 1, 2)$ is also entire and the equation (2.1) is also valid for any complex $t$. Moreover, we have the ridge property ([6], Theorem 7.1.2) which can be written, since $f_j$ is evidently without zeros,

(2.2) $$u_j(0, y) - u_j(x, y) \geq 0 \quad (j = 1, 2)$$

for any real $x$ and $y$ where $u_j$ is defined by

$$u_j(x, y) = \text{Re} \log f_j(x + iy).$$

From the Theorem 1 of the introduction, it follows that for any complex $t$

(2.3) $$f_1(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{p} \sum_{k=1}^{r_j} l'_{j,k} [\exp (ik\alpha_{j,k} t) - 1] \right\},$$

where $c$ and the $l'_{j,k}$ are real constants and where $s_j$ is defined by

$$s_j\alpha_{j,1} \leq \sup_k \alpha_{k,r_k} < (s_j + 1)\alpha_{j,1}.$$

From (2.3), it follows by an elementary computation that

(2.4) $$u_1(0, y) - u_1(x, y) = 2 \sum_{j=1}^{p} \sum_{k=1}^{r_j} l'_{j,k} \sin^2 \left( \frac{1}{2}k\alpha_{j,k} x \right) \exp (k\alpha_{j,k} y).$$

We show now by induction that all the $l'_{j,k}$ for $k\alpha_{j,k} \notin \{ \alpha_{j,k} \}$ are
equal to zero and that all the $l_{j,k}$ for $k\alpha_{j,t} \in \{\alpha_{j,k}\}$ are nonnegative. (It is sufficient to show that all the $l_{j,k}$ are nonnegative since if $l_{j,k}$ for $k\alpha_{j,t} \in \{\alpha_{j,k}\}$ is nonnegative, the corresponding term in $f_2$ is also nonnegative and their sum is zero).

First of all, we show that

\begin{equation}
(2.5) \quad l_{j,t} \geq 0 .
\end{equation}

Indeed, from Kronecker's theorem ([3], Theorem 444), it is possible to find $x = x(y)$ such that

\begin{equation}
(2.6) \quad \sin \left( \frac{1}{2} k \alpha_{j',x} \right) = o(\exp \left[ - \frac{1}{2} s_j \alpha_{j',y} \right]) \quad (y \to \infty) \quad j' \neq j, k = 1, \ldots, s_j
\end{equation}

and

\begin{equation}
(2.7) \quad \sin \left( \frac{1}{2} s_j \alpha_{j,t} x \right) \geq 1 - \varepsilon .
\end{equation}

We have then from (2.4)

\begin{equation*}
u_i(0, y) - u_i(x, y) = O[l_{j,t} \exp (s_j \alpha_{j,y})]
\end{equation*}

when $y \to \infty$ and (2.2) implies (2.5).

Let now $k < s_j$ and let $\nu$ be the smallest integer greater than $k$ such that $l_{j,\nu} > 0$ (if such a $\nu$ does not exist, the preceding proof is still valid). From the hypothesis of induction, we can suppose that $l_{j,\nu}$ is zero if $k' (> k)$ is not a multiple of $\nu$. From Kronecker's theorem, it is possible to find $x = x(y)$ and an integer $p_j$ such that (2.6) and

\begin{equation}
(2.8) \quad x \alpha_{j,t} - 2p_j \pi - \frac{2\pi}{\nu} = o(\exp \left[ - \frac{1}{2} s_j \alpha_{j,y} \right]) \quad (y \to \infty)
\end{equation}

are satisfied. We have then from (2.4)

\begin{equation*}
u_i(0, y) - u_i(x, y) = O[l_{j,t} \sin^2 \left( \frac{1}{2} k \alpha_{j,t} x \right) \exp (k \alpha_{j,y})]
\end{equation*}

and

\begin{equation*}
\sin^2 \left( \frac{1}{2} k \alpha_{j,t} x \right) \geq c > 0 .
\end{equation*}

It follows from (2.2) that

\begin{equation*}
l_{j,t} \geq 0
\end{equation*}

and the theorem is demonstrated.

We can generalize the Theorem 3 in the following manner:

**Theorem 4.** Let $f_3$ be the infinitely divisible characteristic function of the variable $t$ defined by

\begin{equation*}
f_3(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{p} \sum_{k=1}^{s_j} \lambda_{j,k} [\exp (i\alpha_{j,k} t) - 1] + \sum_{q=1}^{\infty} \mu_q [\exp (i\beta_q t) - 1] \right\} ,
\end{equation*}
where the following conditions are satisfied

1. \( \gamma \) is a real constant;
2. the \( \lambda_{j,k} \) and the \( \mu_q \) are nonnegative constants and there exists a positive constant \( K \) such that
   \[ \mu_q = O[\exp(-K\beta_q^2)] \quad (q \to \infty) \cdot \]
3. the \( \alpha_{j,k} \) and the \( \beta_q \) are positive constants such that
   - (a) \( \alpha_{j,k+1}/\alpha_{j,k} \) \( (k = 1, \ldots, r_j - 1; j = 1, \ldots, p) \) and \( \beta_{q+1}/\beta_q \) \( (q = 1, 2, \ldots) \) are integers greater than 1;
   - (b) \( \alpha_{i,1}, \ldots, \alpha_{p,1} \) and \( \beta_1 \) are rationally independent. If \( f_i \) is a factor of \( f_0 \), then
   \[ f_i(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{r_j} \sum_{k=1}^{r_j} l_{j,k} [\exp(i\alpha_{j,k} t) - 1] + \sum_{q=1}^{\infty} m_q [\exp(i\beta_q t) - 1] \right\}, \]
where \( c \) is a real constant and the \( l_{j,k} \) and the \( m_q \) satisfy
   \[ 0 \leq l_{j,k} \leq \lambda_{j,k}; \quad 0 \leq m_q \leq \mu_q. \]

**Proof.** The proof is essentially the same as the preceding. Using the Theorem 2 of the introduction, we obtain the representation
   \[ f_i(t) = \exp \left\{ i\gamma t + \sum_{j=1}^{r_j} \sum_{k=1}^{r_j} l'_{j,k} [\exp(i\alpha_{j,k} t) - 1] \right\}, \]
where \( c, \) the \( l'_{j,k} \) and the \( m'_q \) are real constants, the \( m_q \) satisfy
   \[ 0 \leq m_q \leq \mu_q \]
and where \( s_j, \sigma \) and \( \tau \) are defined by \( (d = \sup_j \alpha_{j,r_j}) \)
   \[ s_j \alpha_{j,1} \leq d < (s_j + 1) \alpha_{j,1}, \]
   \[ \sigma \beta_{1} \leq d < (\sigma + 1) \beta_{1}, \]
   \[ \beta_{\tau-1} \leq d < \beta_\tau. \]

The proof of the nonnegativity of all the \( l'_{j,k} \) and of all the \( m'_q (q \leq \sigma) \) (which implies that all the \( l'_{j,k} \) for \( k \alpha_{j,k} \in \{ \alpha_{j,k} \} \) and all the \( m'_q \) for \( q \beta_q \in \{ \beta_q \} \) are zero) is the same except that we use instead of the Theorem 444 of [3]) the other form of Kronecker's theorem (Theorem 443 of [3]) which asserts that the values of \( z \) satisfying (2.6) and (2.7) (or (2.6) and (2.8)) can be taken in the form \( 2\kappa \pi/\beta_q \) (\( \kappa \) integer).

3. Some auxiliary results. We enumerate now some results which are useful in the following section.
LEMMA 1. ([7], Corollary of the Theorem 1). Let $f$ be a function of the complex variable $z$, analytic in the half-plane $\text{Re } z \geq 0$ and satisfying the conditions

(1) $|f(z)| \leq M_1 |z + 1|^a$ for $\text{Re } z = 0$,

(2) $|f(z)| \leq M_2 |z + 1|^b \exp (b \text{Re } z)$ for $\text{Im } z = 0$,

(3) $|f(z)| \leq M_3 |z + 1|^c \exp [d(\text{Re } z)^2]$ for $\text{Re } z \geq 0$,

where $M_1, M_2, M_3$ are positive constants and $a, b, c$ ($\geq a$) and $d$ are nonnegative constants. Then in all the half-plane $\text{Re } z \geq 0$

$$|f(z)| \leq M_1 |z + 1|^a \exp (b \text{Re } z).$$

LEMMA 2. Let $f$ be a function of the $n$ complex variables $z = (z_1, \cdots, z_n)$ admitting the representation

$$f(z) = \sum_{p_1=0}^{\infty} \cdots \sum_{p_n=0}^{\infty} d_{p_1,\cdots, p_n} \exp \left( 2\pi i \sum_{j=1}^{n} \frac{p_j z_j}{T_j} \right),$$

where $T_j > 0$ ($j = 1, \cdots, n$). In order that the constants $d_{p_1,\cdots, p_n}$ satisfy for some $K > 0$ the relation

$$d_{p_1,\cdots, p_n} = O\left( \exp \left( -K \sum_{j=1}^{n} p_j^2 \right) \right) \quad \left( \sum_{j=1}^{n} p_j \to \infty \right),$$

it is necessary that $f$ be an entire function satisfying

$$\ln |f(z)| = O\left( \sum_{j=1}^{n} |\text{Re } z_j|^2 \right) \quad (|z| \to \infty)$$

and sufficient that $f$ be an entire function satisfying

$$\ln |f(z)| = O\left( \sum_{j=1}^{n} (|\text{Re } z_j|^2 + \ln |z|) \right)$$

$$\left( |z| \to \infty \right) \quad \left( |z|^2 = \sum_{j=1}^{n} |z_j|^2 \right).$$

In the case $n = 1$, this lemma is a particular case of the Theorem 2 of [7]. The proof in the general case is the same as in [7] and is therefore omitted.

LEMMA 3. If the entire function $f$ of the variable $z$ satisfies for some real $K$ the condition

$$|f(z)| \leq \exp \left[ K \text{Re } z + O(\ln |z|) \right] \quad (\text{Re } z \geq 0)$$

when $|z| \to \infty$ and admits an expansion of the form

$$f(z) = \sum_{p=-\infty}^{+\infty} a_p \exp \left( \frac{2\pi p z}{T} \right),$$
where $T > 0$ and the series converges uniformly in every bounded set, then
\[ a_p = 0 \]
for
\[ p > [KT(2\pi)^{-1}] . \]

This lemma is a particular case of the Theorem 3 of [7].

**Lemma 4.** If $\varphi$ is an entire function of the $n$ variables $z = (z_1, \cdots, z_n)$ such that for any $x, y \in \mathbb{R}^n$ and any $\varepsilon > 0$
\[
\begin{align*}
    u(x, y) &= \mathfrak{Re} \varphi(x + iy) = O[\exp (\tau + \varepsilon)(|x| + |y|)] \\
               &\quad (|x| + |y| \to -\infty),
\end{align*}
\]
then $\varphi$ is a function of exponential type $\tau$ with respect to the hermitian norm.

This lemma is a particular case of the Lemma 2 of the theorem 2.5 of [1].

For the following lemma, we recall the

**Definition.** Let $\{f_n\}$ be a sequence of functions belonging to a Banach space of functions. The $f_n$ are said topologically independent if the relation
\[
\lim_{\varepsilon \to 0} \left\| \sum_{n=1}^{\infty} a_n(\varepsilon)f_n \right\| = 0
\]
implies
\[
\lim_{\varepsilon \to 0} a_n(\varepsilon) = 0 \quad n = 1, 2, \cdots.
\]

We have then the

**Lemma 5.** (Lemma 1 of the Theorem 6.1 of [1]). Let $\{\lambda_j\}$ a sequence of real numbers such that
\[
\sum_{j=1}^{\infty} \frac{1}{|\lambda_j|} < +\infty.
\]
Then the functions $1, z, \exp(\lambda_j z)$ ($j = 1, 2, \cdots$) are topologically independent in the space $C(a, b)$ of continuous functions $f$ on $[a, b]$ $(-\infty < a < b < +\infty)$ with the norm $\|f\| = \sup_{a < z < b} |f(z)|$.

Recall ([1], Chapter 4) that a function $\varphi$ of the $n$ complex variables $z = (z_1, \cdots, z_n)$ is said a ridge function if it is an entire function satisfying for any $z \in C^n$ the relation
\[
|\varphi(z)| \leq \varphi(\mathfrak{Re} z) \quad (\mathfrak{Re} z = (\mathfrak{Re} z_1, \cdots, \mathfrak{Re} z_n)).
\]
We have the

**Lemma 6.** Let $\varphi_0$ be a ridge function of the $n$ variables $z = (z_1, \ldots, z_n)$ without zeros and $\varphi_1$ and $\varphi_2$ be two ridge functions such that

$$\varphi_0 = \varphi_1 \varphi_2.$$

There exists a positive constant $C$ such that

$$M(r; \log \varphi_j) \leq 6rM(r + 1; \log \varphi_0) + Cr(r + 1) \quad (j = 1, 2),$$

where

$$M(r; f) = \sup_{|z| = r} |f(z)|.$$

**Proof.** Let

$$\psi_j(z) = \log [\varphi_j(z)], \quad \text{Re} \psi_j(x + iy) = u_j(x, y)$$

for any $x, y \in \mathbb{R}^n$ ($j = 0, 1, 2$). Since $\varphi_1$ and $\varphi_0/\varphi_1$ are ridge functions without zeros, we have

$$0 \leq u_1(x, 0) - u_2(x, y) \leq u_0(x, 0) - u_0(x, y) \leq 2M(r; \psi_0)$$

for $|x + iy| \leq r$.

We estimate now $|u_0(x, 0)|$. For that, we use the existence for any ridge function $\varphi$ of a positive constant $C_\varphi$ such that

$$\log \varphi(x) \geq -C_\varphi |x|$$

for any $x \in \mathbb{R}^n$. Indeed, since $\log \varphi(\lambda \theta)$ is for any direction $\theta$ of $\mathbb{R}^n$ a convex function of $\lambda$, we have

$$\log \varphi(\lambda \theta) \geq \log \varphi(0) + \lambda (\alpha \cdot \theta),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_\varphi(0) = \{\partial \varphi(0)/\partial z_j \}$ and where $(\alpha \cdot \theta)$ indicates the scalar product of the vectors $\alpha$ and $\theta$. The relation (3.2) is an immediate consequence of (3.3).

From (3.2), we have

$$u_0(x, 0) = \psi_0(x) \geq -C_1 |x|,$$

$$u_0(x, 0) = \psi_0(x) - \psi_2(x) \leq \psi_0(x) + C_2 |x|.$$

From (3.4) and (3.5), it follows

$$|u_1(x, 0)| \leq M(r; \psi_0) + Cr \quad (|x| \leq r)$$

for some positive constant $C$ and from (3.1)

$$|u_1(x, y)| \leq 3M(r; \psi_0) + Cr (|x + iy| \leq r).$$

Let now $g_\theta$ the function of the complex variable $\lambda$ defined by
for some direction $\theta$ of $\mathbb{R}^n$. Then
\[
v_\theta(\mu, \nu) = \text{Re} \, g_\theta(\mu + i\nu) = u_\theta(\mu \theta + i\nu \theta)
\]
for any real $\mu$ and $\nu$. We have then $(\lambda = \mu + i\nu)$
\[
g_\theta'(\lambda) = \frac{1}{\pi} \int_0^{2\pi} v_\theta(\mu + \cos \alpha, \nu + \sin \alpha) e^{-i\alpha} d\alpha;
\]
so that
\[
|g_\theta'(\lambda)| \leq 2 \sup_{0 \leq \xi \leq 2\pi} |v_\theta(\mu + \cos \alpha, \nu + \sin \alpha)|
\]
\[
\leq 6M(r + 1; \log \varphi_0) + C(r + 1)
\]
(\(|\lambda| \leq r\)) for some positive constant $C$. Since
\[
g_\theta(\lambda) = \int_0^\pi g_\theta'(\xi) d\xi + \frac{1}{i} \int_0^\pi g_\theta'(\mu + i\nu) d\eta
\]
and since $\theta$ is arbitrary, the lemma is a consequence of (3.6).

**Lemma 7.** (Lemma of the Theorem 5 of [2]). If $f$ is an entire characteristic function of the two variables $t_1$ and $t_2$, and $t_2$ a real constant, the function $f_{t_2}$ defined by
\[
f_{t_2}(t_1) = \frac{f(t_1, it_2)}{f(0, it_2)}
\]
is an entire characteristic function.

4. The case of several variables. First of all, we consider the case of functions of two variables.

**Theorem 5.** Let $f_\circ$ be the infinitely divisible characteristic function of the two variables $t = (t_1, t_2)$ defined by
\[
f_\circ(t) = \exp \left\{ i\pi(t) + \sum_{j=1}^\infty (\lambda_j \exp(i\alpha_j t_1) - 1) + \mu_j [\exp(i\beta_j t_2) - 1] + \nu_j [\exp(i\alpha_j t_1 + i\beta_j t_2) - 1] \right\},
\]
where the following conditions are satisfied
\begin{enumerate}
  \item $\pi$ is an homogeneous polynomial of degree one with real coefficients;
  \item $\lambda_j, \mu_j, \nu_j$ are nonnegative constants and there exists a positive constant $K$ such that
\end{enumerate}
\[ \lambda_j = O(\exp(-K\alpha_j)); \quad \mu_j = O(\exp(-K\beta_j)); \]
\[ \nu_j = O(\exp(-K(\alpha_j^2 + \beta_j^2))) \quad (j \to +\infty); \]

(3) the \( \alpha_j \) are positive constants satisfying the three conditions

(a) there exists \( q \) such that \( \alpha_{j+1}/\alpha_j \) is an integer greater than 1 for \( j \geq q; \)

(b) the set \{\( \alpha_j; j < q \)\} can be decomposed in \( p \) sets \{\( \alpha_j,k \)\} \( (j = 1, \ldots, p; k = 1, \ldots, r_j; \sum_{j=1}^p r_j = q - 1) \) such that \( \alpha_{j,k+1}/\alpha_{j,k} \) \( (k = 1, \ldots, r_j - 1; j = 1, \ldots, p) \) is an integer greater than 1 and \( \alpha_{1,1}, \ldots, \alpha_{p,1} \) are rationally independent;

(c) either \( \alpha_{q,1} \) is a multiple of one of the \( \alpha_{j,r_j} \) or \( \alpha_{1,1}, \ldots, \alpha_{p,1} \)
and \( \alpha_{q,1} \) are rationally independent;

(4) the \( \beta_j \) are positive constants having the same property.

If \( f_1 \) is a factor of \( f_0 \), then

\[ f_1(t) = \exp \left\{ iP(t) + \sum_{j=1}^\infty (l_j[\exp(ia_j t_j) - 1] + m_j[\exp(ib_j t_j) - 1] + n_j[\exp(ia_j t_j + ib_j t_j) - 1]) \right\} \]

where \( P \) is an homogeneous polynomial of degree one with real coefficients and where \( l_j, m_j, n_j \) are constants satisfying the conditions

\[ 0 \leq l_j \leq \lambda_j; \quad 0 \leq m_j \leq \mu_j; \quad 0 \leq n_j \leq \nu_j. \]

Proof. Let \( f_1 \) and \( f_2 \) be the two characteristic functions such that for any real \( t_1 \) and \( t_2 \)

\[ f_0(t_1, t_2) = f_1(t_1, t_2)f_2(t_1, t_2). \]

Since \( f_0 \) is an entire characteristic function, from Raikov's theorem ([1], Theorem 2.3), \( f_j \) is also entire \( (j = 1, 2) \) and the equation (4.1) is also valid for any complex \( t_1 \) and \( t_2 \). Letting

\[ \varphi_j(x) = f_j(-iz), \]
\[ u_j(x, y) = \text{Re} \log \varphi_j(x + iy), \]

\( j = 0, 1, 2 \) for any \( x, y \in \mathbb{R}^2 \), since \( \varphi_j \) is a ridge function ([1], Corollary 1 of the Theorem 2.1), we have

\[ 0 \leq u_1(x, 0) - u_1(x, y) \leq u_0(x, 0) - u_0(x, y) \]

for any \( x, y \in \mathbb{R}^2 \).

If we fix \( z \) real, using the Lemma 7 and the Theorem 4, we have

\[ \log \varphi(z) = a + bz + \sum_{j=1}^\infty c_j \exp(\alpha_j z_j), \]

where \( a, b, c_j \) are functions of \( z \), real for \( z \) real and satisfying
(4.4) \[ 0 \leq c_j(z_2) \leq \lambda_j + \nu_j \exp(\beta_j z_2) . \]

If we fix \( z_1 \) real, we have

(4.5) \[ \log \varphi_i(z) = r + sz_1 + \sum_{j=1}^{\infty} t_j \exp(\beta_j z_2) , \]

where \( r, s, t_j \) are functions of \( z_1 \), real for \( z_1 \) real and satisfying

(4.6) \[ 0 \leq t_j(z_1) \leq \mu_j + \nu_j \exp(\alpha_j z_1) . \]

From (4.3) and (4.5), we obtain the equation for any real \( z_1 \) and \( z_2 \)

\[ a + b z_1 + \sum_{j=1}^{\infty} c_j \exp(\alpha_j z_1) = r + s z_1 + \sum_{k=1}^{\infty} t_k \exp(\beta_k z_2) \]

which can be solved by using the Lemma 5 (for the details, see the proof of the Theorem 6.1 of [1]). We obtain for any \( z_1 \) and \( z_2 \) complex the representation

(4.7)

\[ \log \varphi_i(z) = c + P(z) + dz_1 z_2 + \sum_{j=0}^{\infty} [\rho_j z_2 \exp(\alpha_j z_1) + \sigma_j z_1 \exp(\beta_j z_2)] + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} n_{j,k} \exp(\alpha_j z_1 + \beta_k z_2) , \]

where all the constants and the coefficients of the homogeneous polynomial \( P \) of degree one are real (with the convention \( \alpha_0 = \beta_0 = n_{0,0} = 0 \)). By an elementary computation, we obtain

\[ u_t(x, 0) - u_t(x, y) = dy_1 y_2 + \sum_{j=1}^{\infty} [2 \rho_j x_2 \exp(\alpha_j x_1) \sin^2(\frac{1}{2} \alpha_j y_1) + 2 \sigma_j x_1 \exp(\beta_j x_2) \sin^2(\frac{1}{2} \beta_j y_2) + \rho_j y_2 \exp(\alpha_j x_1) \sin(\alpha_j y_1) + \sigma_j y_1 \exp(\beta_j x_2) \sin(\beta_j y_2)] + 2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} n_{j,k} \exp(\alpha_j x_1 + \beta_k x_2) \sin^2(\frac{\alpha_j y_1 + \beta_k y_2}{2}) . \]

Letting \( |y_1| \to \infty \), we obtain from (4.2) and (4.8)

\[ dy_2 + \sum_{j=1}^{\infty} \sigma_j \exp(\beta_j x^2) \sin(\beta_j y_2) = 0 . \]

Since the expression in the left member is the imaginary part of

\[ dz_2 + \sum_{j=1}^{\infty} \sigma_j \exp(\beta_j z_2) , \]

we deduce from the Lemmas 4 and 3 that

\[ d = \sigma_j = 0 . \]
In the same manner, letting $|y_2| \to \infty$, we obtain
\[ \rho_j = 0. \]

From the Lemma 5 and (4.4), it follows that for any real $x_2$
\[ 0 \leq \sum_{k=0}^{\infty} n_{j,k} \exp (\beta_k x_2) \leq \lambda_j + \nu_j \exp (\beta_j x_2). \]

On the other hand, $\log \varphi_0$ satisfies
\[ \log \varphi_0(z) = O[|z| (1 + \exp (N |\Re z|^2))] \quad (|z| \to \infty) \]
for some $N > 0$. It follows from the lemma 6 that
\[ \log \varphi_i(z) = O[|z|^2 (1 + \exp (N |\Re z|^2))] \quad (|z| \to \infty) \]
and from the sufficient part of the Lemma 2 applied to
\[ \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} n_{j,k} \exp (\alpha_j z_1 + \beta_k z_2) \]
(the constant $q_j$ is defined in the statement of the theorem and the constant $q_2$ is the analogous for the $\beta_k$), we have for some $\kappa' > 0$
\[ n_{j,k} = O[\exp (-\kappa'(j^2 + k^2))] \quad (|j| + |k| \to \infty), \]
that implies from the necessary part of the Lemma 2
\[ \sum_{k=0}^{\infty} n_{j,k} \exp (\beta_k z_2) = O[\exp (N' |\Re z_2|^2)] \quad (|z_2| \to \infty) \]
for any complex $z_2$ and some $N' > 0$. Using the Lemma 1, we obtain
\[ \sum_{k=0}^{\infty} n_{j,k} \exp (\beta_k z_2) = O[\exp (\beta_j z_2)] \quad (|z_2| \to \infty) \]
in $\{\Re z_2 \geq 0\}$, that implies from the Lemma 3
\[ n_{j,k} = 0 \]
for all the $k$ such that $\beta_k > \beta_j$ and from (4.9)
\[ n_{j,j} \geq 0, \quad n_{j,0} \geq 0. \]

In the same manner, from (4.6), we obtain
\[ n_{j,k} = 0 \]
for all the $j$ such that $\alpha_j > \alpha_k$ and
\[ n_{0,j} \geq 0. \]
In particular, we have $\left( q = \sup (q_1, q_2) \right)$
\[ n_{j,k} = 0 \]

if \((j, k) \in \{(j, j), (0, j), (j, 0)\}\) and either \(j \geq q\) or \(k \geq q\).

(4.7) becomes (with \(l_j = n_{j,0}, m_j = n_{0,j}, n_j = n_{j,j}\) if \(j \geq q\)).

\[
\log \varphi_\ell(z) = c + P(z) \\
+ \sum_{j=q}^{\infty} [l_j \exp(\alpha_j z) + m_j \exp(\beta_j z) + n_j \exp(\alpha_j z_1 + \beta_j z_2)] \\
+ \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} n_{j,k} \exp(\alpha_j z_1 + \beta_k z_2)
\]

and (4.8) becomes

\[
\frac{1}{2} [u_\ell(x, 0) - u_\ell(x, y)] \\
= \sum_{j=q}^{\infty} \left[ l_j \exp(\alpha_j x_1) \sin^2 \left( \frac{1}{2} \beta_j y_1 \right) + m_j \exp(\beta_j x_1) \sin^2 \left( \frac{1}{2} \beta_j y_2 \right) \\
+ n_j \exp(\alpha_j x_1 + \beta_j x_2) \sin^2 \left( \frac{\alpha_j y_1 + \beta_j y_2}{2} \right) \right] \\
+ \sum_{j=0}^{q-1} \sum_{k=0}^{q-1} n_{j,k} \exp(\alpha_j x_1 + \beta_k x_2) \sin^2 \left( \frac{\alpha_j y_1 + \beta_k y_2}{2} \right).
\]

We show now by induction that all the \(n_{j,k}\) \((i \leq q - 1, k \leq q - 1)\) are nonnegative (that implies \(n_{j,k} = 0\) if \((j, k) \not\in \{(j, j), (j, 0), (0, j)\}\)).

We show that this result is true for \(j = j_0\) such that \(\alpha_{j_0} = \sup_{j=1,\ldots,q} \alpha_j\).

We put \(y_1 = 2\pi/\alpha_q\) and choose \(y_2\) from Kronecker’s theorem (Theorem 443 of [3]) such that

(a) \(y_2 = \frac{2\kappa' \pi}{\beta_{\ell'}}\)

(\(\kappa'\) integer) where \(\beta_{\ell'}\) is the smallest number greater than \(\beta_k\) such that \(\beta_{\ell'}/\beta_{\ell'}\) is integer;

(b) \(\sin \left( \frac{\alpha_{j_0} y_1 + \beta_{\ell'} y_2}{2} \right) = o[\exp(\frac{1}{2} \beta_{\ell'} x_2)]\) \((x_2 \to \infty)\)

for all the \(\ell'\) such that \(\beta_{\ell'} \geq \beta_k\);

(c) \(\sin \left( \frac{\alpha_{j_0} y_1 + \beta_{\ell'} y_2}{2} \right) \geq C > 0\).

Then if \(x_3\) is chosen great enough, we obtain from (4.11)

\[
u_\ell(x, 0) - u_\ell(x, y) = O \left[ n_{j_0,k} \exp(\alpha_{j_0} x_1 + \beta_k x_2) \sin^2 \left( \frac{\alpha_{j_0} y_1 + \beta_k y_2}{2} \right) \right]
\]

\((x_1 \to \infty)\), that implies with (4.2)

\[ n_{j_0,k} \geq 0 \].
Let now \((j, k)\) arbitrary. We can suppose that
\[ n_{j, k'} \geq 0 \quad \text{if} \quad (j', k') \in \{(j', j'), (j', 0), (0, j')\} \]
and
\[ n_{j, k'} = 0 \quad \text{if} \quad (j', k') \in \{(j', j'), (j', 0), (0, j')\} \]
if either \(\alpha_j > \alpha_j\) or \(j' = j\), \(\beta_{k'} > \beta_k\). Then we choose \(y_i\) from Kronecker’s theorem such that
\[
(a) \quad y_i = \frac{2\kappa \pi}{\alpha_i}
\]
\((\kappa\text{ integer})\) where \(\alpha_i\) is the smallest integer greater than \(\alpha_j\) such that \(\alpha_i/\alpha_j\) is integer;

\[
(b) \quad \sin\left(\frac{1}{2}\alpha_j y_i\right) = o[\exp\left(-\frac{1}{2}\alpha_j x_i\right)] \quad (x_i \to \infty)
\]
for all \(j'\) such that \(\alpha_{j'} > \alpha_j\);

\[
(c) \quad \left| \sin\left(\frac{1}{2}\alpha_j y_i\right) \right| \geq c > 0
\]
We choose now \(y_z\) such that, from Kronecker’s theorem
\[
(\alpha) \quad y_z = \frac{2\kappa' \pi}{\beta_z}
\]
\((\kappa'\text{ integer})\) where \(\beta_z\) is the smallest integer greater than \(\beta_k\) such that \(\beta_z/\beta_k\) is integer;

\[
(\beta) \quad \sin\left(\frac{\alpha_j y_z + \beta_{j'} y_z}{2}\right) = o[\exp\left(-\frac{1}{2}\alpha_j x_i\right)] \quad (x_i \to \infty)
\]
for all \(j'\) such that \(\alpha_{j'} > \alpha_j\);

\[
(\gamma) \quad \sin\left(\frac{\alpha_j y_z + \beta_{j'} y_z}{2}\right) = o[\exp\left(-\frac{1}{2}\alpha_j x_i\right)] \quad (x_i \to \infty)
\]
if \(\beta_j > \beta_k\) (otherwise, this condition is superfluous);

\[
(\delta) \quad \left| \sin\left(\frac{\alpha_j y_z + \beta_{j'} y_z}{2}\right) \right| \geq C' \geq 0
\]
We have then, from (4.11), if \(x_z\) is chosen great enough,
\[
u_i(x, 0) - u_i(x, y) = O\left[ n_{j, k} \exp\left(\alpha_j x_i + \beta_k x_z\right) \sin^2\left(\frac{\alpha_j y_z + \beta_{j'} y_z}{2}\right) \right]
\]
\((x_i \to \infty)\), that implies
\[ n_{j, k} \geq 0 \]
and the theorem is demonstrated, the value of $c$ in (4.10) being determined by the condition $\log \varphi_c(0) = 0$.

From this theorem, we deduce easily by the method of the Chapters 5 and 6 of [1] the

**THEOREM 6.** Let $f_0$ be the infinitely divisible characteristic function of the $n$ variables $t = (t_1, \cdots, t_n)$ defined by

$$f_0(t) = \exp \left\{ i\pi(t) + \sum_{j=1}^n \lambda_{j,c} \left[ \exp \left( i \sum_{k=1}^n \varepsilon_k \alpha_{j,k} t_k \right) - 1 \right] \right\},$$

where the following conditions are satisfied:

1. $\pi$ is an homogeneous polynomial of degree one with real coefficients;
2. $\varepsilon_k = 0$ or 1 and $\sum_k$ indicates the summation on the $2^n - 1$ values of $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ different from $(0, \cdots, 0)$;
3. $\lambda_{j,c}$ are nonnegative constants and there exists a positive constant $K$ such that
   $$\lambda_{j,c} = O\left[ \exp \left( -K \sum_{k=1}^n \varepsilon_k \alpha_{j,k}^3 \right) \right] \quad (j \to +\infty);$$
4. $\{\alpha_{j,k}\}$ is, for $k = 1, \cdots, n$, a sequence of positive numbers satisfying the condition (3) of the Theorem 5.

If $f_i$ is a factor of $f_0$, then

$$f_i(t) = \exp \left\{ iP(t) + \sum_{j=1}^n \lambda_{j,c} \left[ \exp \left( i \sum_{k=1}^n \varepsilon_k \alpha_{j,k} t_k \right) - 1 \right] \right\},$$

where $P$ is an homogeneous polynomial of degree one with real coefficients and $\lambda_{j,c}$ are constants satisfying the conditions

$$0 \leq l_{j,c} \leq \lambda_{j,c}.$$

With the same method, we can deduce from the Theorem 1' of Ostrovskiy [8] the

**THEOREM 7.** Let $f_0$ be the infinitely divisible characteristic function of the $n$ variables $t = (t_1, \cdots, t_n)$ defined by

$$f_0(t) = \exp \left\{ i\pi(t) + \sum_{j=1}^n \lambda_{j,c} \left[ \exp \left( i \sum_{k=1}^n \varepsilon_k \alpha_{j,k} t_k \right) - 1 \right] \right\},$$

where, beyond the conditions (1), (2), (3) of the preceding theorem, the following condition is satisfied:

1. $\{\alpha_{j,k}\}$ is for $k = 1, \cdots, n$ a sequence of increasing positive numbers such that
2. there exists $q_k$ such that $\alpha_{j+1,k}/\alpha_{j,k} (j \geq q_k)$ is an integer
greater than 1;
(b) there exists a positive constant $a_k$ such that $a_k \leq \lambda_{j,k} \leq 2a_k$
($j < q_k$).
If $f_i$ is a factor of $f_o$, then

$$f_i(t) = \exp \left\{ iP(t) + \sum_{j=1}^{\infty} \sum_{v} l_{j,v} \left[ \exp \left( i \sum_{k=1}^{n} \varepsilon_k \alpha_{j,v,t_k} \right) - 1 \right] \right\},$$

where $P$ is an homogeneous polynomial of degree one with real coefficients and $l_{j,v}$ are constants satisfying the conditions

$$0 \leq l_{j,v} \leq \lambda_{j,v}.$$ 

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