

# Pacific Journal of Mathematics

**NONOSCILLATORY SOLUTIONS OF SECOND ORDER  
NONLINEAR DIFFERENTIAL EQUATIONS**

LYNN HARRY ERBE

## NONOSCILLATORY SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

LYNN H. ERBE

We consider here a generalization of the equation

$$x'' + a(t)x^{2n+1} = 0$$

where  $a(t)$  is a continuous non-negative function on  $[0, +\infty)$  and  $n \geq 0$  is an integer. Necessary and sufficient conditions are given for the existence of

- (1) a bounded nonoscillatory solution with prescribed limit at  $\infty$ ;
- (2) a nonoscillatory solution whose derivative has a positive limit at  $\infty$ .

Specifically, we are concerned with the asymptotic behavior of the solutions of the following second order nonlinear differential equation :

$$(1) \quad x'' + f(t, x)g(x') = 0 .$$

We shall assume the following conditions hold :

$$(A_0) \quad f(t, x), g(x'), \text{ and the partial derivative function } f_x(t, x) \text{ are all continuous for } t \geq 0, x' \geq 0, \text{ and } |x| < +\infty .$$

$$(A_1) \quad f(t, 0) = 0, t \geq 0 .$$

$$(A_2) \quad f_x(t, x) \geq 0 \text{ and is nondecreasing in } x \text{ for } t \geq 0 \text{ and } x \geq 0 .$$

$$(A_3) \quad g(x') > 0 \text{ for all } x' \geq 0 .$$

As a special case we have the equation

$$(2) \quad x'' + a(t)x^{2n+1} = 0, n \geq 0 ,$$

in which  $a(t) \geq 0$  for  $t \geq 0$  and  $g(x') = 1$  for all  $x'$ . Oscillatory and nonoscillatory properties of (2) for the case  $n \geq 1$  were investigated by Atkinson in [1], Moore and Nehari in [5], and Utz in [9]. Generalizations of equation (2) have been considered by Waltman in [7] and [8], Nehari in [6], Wong in [10], and Macki and Wong in [4].

We shall study equation (1) by considering the equation

$$(3) \quad x'' + f_x(t, \alpha)x = 0 ,$$

where  $\alpha$  is some real constant depending on solutions of (1). To do this we shall need to establish several lemmas concerning the equation

$$(4) \quad x'' + p(t)x = 0,$$

where  $p(t)$  is continuous and satisfies  $p(t) \geq 0$  for  $t \geq 0$ .

LEMMA 1.1. *Let  $[a, b]$  be a compact interval of the reals and suppose there exists a  $\beta(t) \in C^{(2)} [a, b]$  satisfying*

$$\beta(t) > 0, \quad \beta''(t) + p(t)\beta(t) \leq 0, \quad t \in [a, b].$$

*Then  $[a, b]$  is an interval of disconjugacy for equation (4). That is, no nontrivial solution of (4) has more than one zero on  $[a, b]$ .*

*Proof.* If the conclusion is false, then there is a solution  $y(t)$  of (4) satisfying  $y(t_1) = y(t_2) = 0$  and  $y(t) > 0$  on  $(t_1, t_2)$ , where  $a \leq t_1 < t_2 \leq b$ . It follows that there is a  $k > 0$  such that  $ky(t) \leq \beta(t)$  on  $[t_1, t_2]$  and  $ky(t_0) = \beta(t_0)$  for some  $t_1 < t_0 < t_2$ . Therefore,  $ky'(t_0) = \beta'(t_0)$  and for  $t_0 \leq t \leq t_2$  we have

$$ky'(t) - \beta'(t) \geq \int_{t_0}^t -p(s)\{ky(s) - \beta(s)\}ds \geq 0.$$

Hence,

$$ky(t_2) - \beta(t_2) = \int_{t_0}^{t_2} (ky'(s) - \beta'(s))ds \geq 0,$$

which is a contradiction.

REMARK. If there exists an  $\alpha(t) \in C^{(2)} [a, b]$  satisfying

$$\alpha(t) < 0, \quad \alpha''(t) + p(t)\alpha(t) \geq 0, \quad t \in [a, b],$$

then the conclusion of the lemma again holds. (Set  $\beta(t) = -\alpha(t)$ ,  $t \in [a, b]$ .)

Lemma 1.1 is closely related to a theorem of Wintner (see Hartman [2], p. 362, Th. 7.2) and could be obtained directly by setting  $z = \beta'/\beta$ . Also, a function  $\beta(t) \in C^{(2)} [a, b]$  satisfying  $\beta''(t) + p(t)\beta(t) \leq 0$  on  $[a, b]$  is just a special case of an upper solution, as defined by Jackson in [3] for general nonlinear second order differential equations. Likewise  $\alpha(t) \in C^{(2)} [a, b]$  satisfying  $\alpha''(t) + p(t)\alpha(t) \geq 0$  on  $[a, b]$  is a special case of a lower solution.

LEMMA 1.2. *Let  $\alpha(t), \beta(t) \in C^{(2)} [a, b]$  and satisfy  $\alpha''(t) + p(t)\alpha(t) \geq 0$ ,  $\beta''(t) + p(t)\beta(t) \leq 0$ , and  $0 < \alpha(t) \leq \beta(t)$  on  $[a, b]$ . Then for any  $c, d$  with  $\alpha(a) \leq c \leq \beta(a)$ ,  $\alpha(b) \leq d \leq \beta(b)$ , there is a unique solution  $z(t)$  of (4) satisfying  $z(a) = c$ ,  $z(b) = d$ , and  $\alpha(t) \leq z(t) \leq \beta(t)$  on  $[a, b]$ .*

*Proof.* By Lemma 1.1,  $[a, b]$  is an interval of disconjugacy for equation (4) so that the BVP

$$x'' + p(t)x = 0, \quad x(a) = c, \quad x(b) = d$$

has a unique solution  $z(t)$  (see for example [2], p. 351). Since  $z(t)$  cannot have more than one zero on  $[a, b]$  and since initial value problems for (4) have unique solutions, it follows that  $z(t) > 0$  on  $[a, b]$ . If the conclusion of the lemma is false, then assume, to be specific, that  $z(t_1) - \beta(t_1) = z(t_2) - \beta(t_2) = 0$  and  $z(t) > \beta(t)$  on  $(t_1, t_2)$ , where  $a \leq t_1 < t_2 \leq b$ . As in Lemma 1.1, there is a  $k > 0$ ,  $k < 1$ , such that  $0 < kz(t) \leq \beta(t)$  on  $[t_1, t_2]$ , and  $kz(t_0) = \beta(t_0)$ ,  $kz'(t_0) = \beta'(t_0)$  for some  $t_1 < t_0 < t_2$ . Since  $kz(t_2) < z(t_2) = \beta(t_2)$ , this leads to a contradiction as in Lemma 1.1. Hence,  $z(t) \leq \beta(t)$  on  $[a, b]$ . A similar argument shows that  $z(t) \geq \alpha(t)$  on  $[a, b]$  and this proves the lemma.

**LEMMA 1.3.** *Let  $\alpha(t), \beta(t) \in C^{(2)} [a, +\infty)$  with  $\alpha''(t) + p(t)\alpha(t) \geq 0$ ,  $\beta''(t) + p(t)\beta(t) \leq 0$ , and  $0 < \alpha(t) \leq \beta(t)$  on  $[a, +\infty)$ . Then for any  $\alpha(a) \leq c \leq \beta(a)$  there is a solution  $y(t) \in C^{(2)} [a, +\infty)$  of (4) satisfying  $y(a) = c$  and  $\alpha(t) \leq y(t) \leq \beta(t)$  on  $[a, +\infty)$ .*

*Proof.* By Lemma 1.2 for each  $n \geq 1$  there is a solution  $y_n(t) \in C^{(2)} [a, a+n]$  of (4) satisfying  $y_n(a) = c$  and  $\alpha(t) \leq y_n(t) \leq \beta(t)$  on  $[a, a+n]$ . Therefore, for each  $N \geq 1$   $|y_n(t)|$  and hence  $|y_n''(t)|$  are uniformly bounded on  $[a, a+N]$  for all  $n = N$ . Since  $y_n'(t) = y_n'(a) + \int_a^t y_n''(t) dt$ , the  $|y_n'(t)|$  are likewise bounded on  $[a, a+N]$ , uniformly for  $n \geq N$ . Now consider the sequence  $\{y_n(t)\}_{n=1}^\infty$ . By the Ascoli-Arzelà Theorem there is a subsequence  $\{y_n^1(t)\}_{n=1}^\infty$  converging to a solution  $z_1(t)$  of (4) on  $[a, a+1]$ . Inductively, for each  $k \geq 2$  we obtain a subsequence  $\{y_n^k(t)\}_{n=1}^\infty$  of  $\{y_n^{k-1}(t)\}_{n=1}^\infty$  which converges to a solution  $z_k(t)$  of (4) on  $[a, a+k]$ . Therefore, the diagonal sequence  $\{y_n^k(t)\}_{k=1}^\infty$  converges uniformly on each compact subinterval of  $[a, +\infty)$ . That is,

$$z(t) = \lim_{k \rightarrow \infty} y_n^k(t), \quad t \in [a, +\infty),$$

is the desired solution.

2. After these preliminary lemmas, we are now in a position to establish necessary and sufficient conditions for the existence of certain types of solutions of (1).

**THEOREM 2.1.** *Assume  $A_0 - A_3$  hold and let  $\alpha_0 > 0$ . Then the following statements are equivalent:*

(a) *For each  $0 < \alpha < \alpha_0$  there is a solution  $u_\alpha(t)$  of (1) satisfying  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ .*

(b)  $\int_0^\infty t f_y(t, \alpha) dt < +\infty$  for  $0 < \alpha < \alpha_0$ .

*Proof.* (a) implies (b): Assume  $\int_0^\infty t f_y(t, \alpha_1) dt = +\infty$  for some  $0 < \alpha_1 < \alpha_0$  and let  $\alpha_1 < \beta < \alpha_0$ . Let  $u_\beta(t)$  be the corresponding solution of (1) with  $\lim_{t \rightarrow \infty} u_\beta(t) = \beta$ . Let  $\delta > 0$  be such that  $\alpha_1 + \delta < \beta$  and let  $T \geq 0$  be such that  $t \geq T$  implies  $u_\beta(t) \geq \alpha_1 + \delta$ . Then for  $t \geq T$

$$u_\beta'' = -f(t, u_\beta)g(u_\beta') \leq 0$$

so that  $u_\beta'$  decreases to a limit, and this limit clearly must be zero. Therefore,  $u_\beta(t) \leq \beta$  for  $t \geq T$  so that applying the Mean Value Theorem we get

$$\begin{aligned} f_y(t, \alpha_1) &\leq \frac{f(t, u_\beta(t)) - f(t, \alpha_1)}{u_\beta(t) - \alpha_1} \leq \frac{f(t, u_\beta(t))}{u_\beta(t) - \alpha_1} \\ &\leq \frac{u_\beta(t)}{u_\beta(t) - \alpha_1} \frac{f(t, u_\beta(t))}{u_\beta(t)} \leq \frac{\beta}{\delta} \frac{f(t, u_\beta(t))}{u_\beta(t)}, \end{aligned}$$

for  $t \geq T$ . Since  $\lim_{t \rightarrow \infty} u_\beta'(t) = 0$ , there is a  $T_1 \geq T$  such that  $t \geq T_1$  implies  $g(u_\beta'(t)) \geq g(0)/2 > 0$ . Hence, for  $t \geq T_1$  we have

$$w_\beta''(t) = -f(t, u_\beta(t))g(u_\beta'(t)) \leq -k f_y(t, \alpha_1) u_\beta(t),$$

where  $k = g(0)(\delta/2\beta)$ . Also,  $\alpha_1'' = 0 \geq -k f_y(t, \alpha_1) \alpha_1$ . Therefore, by Lemma 1.3 there is a solution  $z(t)$  of the equation

$$(5) \quad x'' + k f_y(t, \alpha_1)x = 0$$

satisfying  $\alpha_1 \leq z(t) \leq u_\beta(t)$  on  $[T_1, +\infty)$ . Let  $w(t) = z(t) \int_{T_1}^t ds/(z(s))^2$  for  $t \geq T_1$ . Then  $w(t)$  is a solution of (5). Since  $z''(t) \leq 0$  for  $t \geq T_1$ , we see that

$$w''(t) = z''(t) \int_{T_1}^t ds/(z(s))^2 \leq 0$$

for  $t \geq T_1$  and hence  $w'(t)$  decreases to a finite nonnegative limit. In fact, we have

$$w'(t) = 1/z(t) + z'(t) \int_{T_1}^t ds/(z(s))^2 \geq 1/z(t) \geq 1/\beta$$

for  $t \geq T_1$ . Hence, for sufficiently large  $t$ , say  $t \geq T_0 \geq T_1$ , we have  $w(t) \geq t/2\beta$ . Therefore, for  $t \geq T_0$  we have

$$\begin{aligned} w'(t) - w'(T_0) &= -k \int_{T_0}^t f_y(s, \alpha_1) w(s) ds \\ &\leq (-k/2\beta) \int_{T_0}^t s f_y(s, \alpha_1) ds \leq 0. \end{aligned}$$

Therefore,

$$w'(T_0) \geq w'(t) + (k/2\beta) \int_{T_0}^t s f_y(s, \alpha_1) ds$$

for  $t \geq T_0$ , so that

$$\int_{T_0}^{\infty} s f_y(s, \alpha_1) ds < +\infty ,$$

which is the desired contradiction.

Conversely, let  $0 < \alpha < \alpha_0$  be given and let

$$M = \max \{g(x') : 0 \leq x' \leq \alpha\} .$$

Let  $T \geq 0$  be such that

$$\int_T^{\infty} (s - T) f_y(s, \alpha) ds < 1/M \text{ and } \int_T^{\infty} f_y(s, \alpha) ds < 1/M .$$

We shall now define a sequence of functions on  $[T, +\infty)$  in the following manner :

Let  $y_0(t) = \alpha$ ,  $t \geq T$ . Now for  $t \geq T$

$$0 \leq \int_t^{\infty} (s - t) f(s, \alpha) g(0) ds \leq \alpha \int_t^{\infty} (s - t) f_y(s, \alpha) g(0) ds \leq \alpha ,$$

so that defining  $y_1(t) = \alpha - \int_t^{\infty} (s - t) f(s, \alpha) g(0) ds$ ,  $t \geq T$ , we have  $0 \leq y_1(t) \leq \alpha$ . Differentiating  $y_1(t)$  we have

$$0 \leq y_1'(t) = \int_t^{\infty} f(s, \alpha) g(0) ds \leq M\alpha \int_t^{\infty} f_y(s, \alpha) ds < \alpha .$$

Proceeding inductively, we define for all  $k \geq 1$

$$y_{k+1}(t) = \alpha - \int_t^{\infty} (s - t) f(s, y_k(s)) g(y_k'(s)) ds , \quad t \geq T ,$$

and obtain  $0 \leq y_k(t)$ ,  $y_k'(t) \leq \alpha$  for all  $k \geq 1$ . It follows that the sequences  $y_k(t)$ ,  $y_k'(t)$ , and  $y_k''(t)$  are uniformly bounded on  $[T, T + n]$  for all  $n \geq 1$ . The Ascoli-Arzelà Theorem and a diagonalization argument yields a subsequence which converges, uniformly on compact subsets of  $[T, +\infty)$ , to a solution  $u_\alpha(t)$  of (1). Obviously,  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ . This completes the proof of the theorem.

REMARK. If  $f(t, x) = -f(t, -x)$  and  $g(x') > 0$  and is continuous for  $|x'| < +\infty$ , then we see that  $\int_t^{\infty} t f_y(t, \alpha) dt < +\infty$  for  $0 < |\alpha| < \alpha_0$  if and only if for each  $0 < |\alpha| < \alpha_0$  there is a solution  $u_\alpha(t)$  of (1) with  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ .

**COROLLARY 2.2.**  $\int_0^\infty t f_y(t, \alpha) dt < +\infty$  for all  $\alpha > 0$  if and only if there is a solution  $u_\alpha(t)$  of (1) with  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$  for all  $\alpha > 0$ .

**COROLLARY 2.3.** If  $f(t, x) = \sum_{i=0}^n a_i(t)x^{2i+1}$  where the  $a_i(t)$  are continuous nonnegative functions for  $t \geq 0$ , then the following statements are equivalent:

(a) There is a solution  $u_\alpha(t)$  of (1) with  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$  for all  $\alpha \neq 0$ .

(b)  $\sum_{i=0}^n \int_0^\infty t a_i(t) dt < +\infty$ .

As examples of equations to which Theorem 2.1 applies but which do not belong to any of the classes of equations considered in references [1], [4] through [8], we have

$$(6) \quad x'' + x(\exp(t(x - \alpha_0)))(1 + x') = 0$$

$$(7) \quad x'' + x(\exp(t(x^2 - \alpha_0^2) + cx'))(1 + (x')^2) = 0,$$

where  $c$  is an arbitrary real number. Then for  $0 < \alpha < \alpha_0$  there is a solution  $u_\alpha(t)$  of (6) with  $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ , and for  $0 < |\alpha| < \alpha_0$  there is a solution  $y_\alpha(t)$  of (7) with  $\lim_{t \rightarrow \infty} y_\alpha(t) = \alpha$ .

3. In [5] it is shown that equation (2) has solutions for which

$$\lim_{t \rightarrow \infty} \frac{y(t)}{t} = \alpha > 0$$

if and only if

$$\int_0^\infty t^{2n+1} a(t) dt < +\infty.$$

In this final section we will show that an analogous result is true for equation (1) provided  $f(t, x)$  satisfies the following additional condition.

(A<sub>4</sub>) There exist real numbers  $c > 0$  and  $\lambda > 0$  such that

$$\liminf_{x \rightarrow \infty} \frac{f(t, x)}{x f_x(t, cx)} \geq \lambda > 0, \text{ for all sufficiently large } t.$$

Note that in the case of equation (2)  $c$  and  $\lambda$  may be any positive real numbers with  $\lambda c^{2n} \leq 1/(2n + 1)$ . We first establish the following lemma.

**LEMMA 3.1.** Assume conditions  $A_0 - A_3$  hold and let there exist a real number  $\beta > 0$  with

$$\int_0^\infty t f_y(t, \beta t) dt < +\infty.$$

Then there exist solutions to (1), say  $y(t)$ , such that  $\lim_{t \rightarrow \infty} y(t)/t$  exists and is positive.

*Proof.* Let  $T > 0$  be such that

$$\int_T^\infty t f_y(t, \beta t) dt < 1/2M,$$

where  $M = \max \{g(x') : 0 \leq x' \leq \beta\}$ . We define a solution of (1) by

$$u(T) = 0, \quad u'(T) = \beta,$$

and we assert that the solution satisfies  $u'(t) \geq \beta/2$  for  $t \geq T$ . Assume, on the contrary, that there is a  $\delta > 0$ ,  $\beta/2 > \delta > 0$ , and a  $t_1 > T$  with  $u'(t_1) = \delta$  and  $u(t) > 0$  on  $(T, t_1]$ . Then for  $T \leq t \leq t_1$  we have

$$(8) \quad u'(T) = u'(t) + \int_T^t f(s, u(s))g(u'(s))ds.$$

Since  $u''(t) \leq 0$  on  $(T, t_1]$  and since  $u(t)$  is concave it follows that

$$\begin{aligned} u'(t) &\leq \beta \quad \text{on } (T, t_1) \quad \text{and} \\ u(t) &\leq \beta(t - T) \quad \text{on } (T, t_1). \end{aligned}$$

Applying the Mean Value Theorem in (8) we have

$$\begin{aligned} \beta = u'(T) &< u'(t) + M\beta \int_T^t s f_y(s, \beta(s - T))ds \\ &\leq u'(t) + M\beta \int_T^t s f_y(s, \beta s)ds < u'(t) + \beta/2. \end{aligned}$$

Hence,  $u'(t_1) > \beta/2$ , a contradiction. Therefore,  $u'(t) \geq \beta/2$  on  $[T, +\infty)$  and hence  $\lim_{t \rightarrow \infty} u'(t)$  exists and is positive which implies that  $\lim_{t \rightarrow \infty} u(t)/t$  exists and is positive.

**THEOREM 3.2.** *Assume conditions  $(A_0) - (A_4)$  hold. Then (1) has solutions, say  $y(t)$ , such that  $\lim_{t \rightarrow \infty} y(t)/t$  exists and is positive if and only if*

$$\int^\infty t f_y(t, \beta t) dt < +\infty \quad \text{for some } \beta > 0.$$

*Proof.* Let  $\alpha > 0$  and let  $y(t)$  be a solution of (1) with

$$\lim_{t \rightarrow \infty} \frac{y(t)}{t} = \alpha.$$

Let  $T \geq 0$  be such that  $t \geq T$  implies  $y(t) \geq \alpha t/2$ . Let

$$m_0 = \min \{g(x') : 0 \leq x' \leq y'(T)\}.$$



By condition  $(A_4)$  there is a  $T_1 \geq T$  such that  $t \geq T_1$  implies

$$f(t, y(t)) \geq \lambda y(t) f_y(t, c\alpha t/2) \geq (kt) f_y(t, c\alpha t/2),$$

where  $k = \lambda\alpha/2$ . Since  $0 < y'(t) \leq y'(T)$  for  $t \geq T$  we have

$$f(t, y(t))g(y'(t)) \geq (m_0 kt) f_y(t, c\alpha t/2), \quad t \geq T_1.$$

Therefore,

$$\begin{aligned} y'(T_1) &= y'(t) + \int_{T_1}^t f(s, y(s))g(y'(s))ds \\ &\geq y'(t) + \int_{T_1}^t (m_0 ks) f_y(s, c\alpha s/2)ds. \end{aligned}$$

Since  $\lim_{t \rightarrow \infty} y'(t) \geq 0$ , this implies that

$$\int_{T_1}^{\infty} s f_y(s, c\alpha s/2)ds < +\infty,$$

and this proves the theorem.

As a simple example of an equation to which the previous theorem applies but which is not considered in references [1], [4] through [8], we have

$$(9) \quad x'' + x^2 (\exp(x - \beta t))(1 + x') = 0,$$

where  $\beta > 0$ . Condition  $(A_4)$  holds for any  $0 < c < 1$  and any  $\lambda > 0$ .

The author wishes to thank Professor Lloyd Jackson and the referee for several helpful comments and suggestions.

#### REFERENCES

1. F. V. Atkinson, *On second order nonlinear oscillations*, Pacific J. Math. **5** (1955), 643-647.
2. P. Hartman, *Ordinary differential equations*, Wiley, 1964.
3. L. K. Jackson, *Subfunctions and second order differential inequalities*, Advances in Math., Academic Press (to appear).
4. J. W. Macki and J. S. W. Wong, *Oscillation of solutions to second order nonlinear differential equations*, Pacific J. Math. **18** (1968) (to appear)
5. R. A. Moore and Z. Nehari, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc. **92-93** (1959), 30-52.
6. Z. Nehari, *On a class of nonlinear second order differential equations*, Trans. Amer. Math. Soc. **94-95** (1960), 101-123.
7. P. Waltman, *Oscillation of solutions of a nonlinear equation*, SIAM Review **5** (1963), 128-130.
8. ———, *Some properties of solution of  $u'' + a(t)f(u) = 0$* , Monat. Math. (1963-64), 50-54.
9. W. R. Utz, *Properties of solutions of  $u'' + g(t)u^{2n-1} = 0$* , Monat. Math. (1961-62), 55-60.

10. J. S. W. Wong, *A note on second order nonlinear oscillation*, SIAM Review **10** (1968), 88-91.

Received June 19, 1968. This research was supported by a NASA Traineeship at the University of Nebraska.

THE UNIVERSITY OF NEBRASKA  
LINCOLN, NEBRASKA, AND  
THE UNIVERSITY OF ALBERTA  
EDMONTON, ALBERTA



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN  
Stanford University  
Stanford, California

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. R. PHELPS  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

# Pacific Journal of Mathematics

Vol. 28, No. 1

March, 1969

Patrick Robert Ahern, <i>On the geometry of the unit ball in the space of real annihilating measures</i> .....	1
Kirby Alan Baker, <i>Equational classes of modular lattices</i> .....	9
E. F. Beckenbach and Gerald Andrew Hutchison, <i>Meromorphic minimal surfaces</i> .....	17
Tae Ho Choe, <i>Intrinsic topologies in a topological lattice</i> .....	49
John Bligh Conway, <i>A theorem on sequential convergence of measures and some applications</i> .....	53
Roger Cuppens, <i>On the decomposition of infinitely divisible probability laws without normal factor</i> .....	61
Lynn Harry Erbe, <i>Nonoscillatory solutions of second order nonlinear differential equations</i> .....	77
Burton I. Fein, <i>The Schur index for projective representations of finite groups</i> .....	87
Stanley P. Gudder, <i>A note on proposition observables</i> .....	101
Kenneth Kapp, <i>On Croisot's theory of decompositions</i> .....	105
Robert P. Kaufman, <i>Gap series and an example to Malliavin's theorem</i> .....	117
E. J. McShane, Robert Breckenridge Warfield, Jr. and V. M. Warfield, <i>Invariant extensions of linear functionals, with applications to measures and stochastic processes</i> .....	121
Marvin Victor Mielke, <i>Rearrangement of spherical modifications</i> .....	143
Akio Osada, <i>On unicity of capacity functions</i> .....	151
Donald Steven Passman, <i>Some <math>5/2</math> transitive permutation groups</i> .....	157
Harold L. Peterson, Jr., <i>Regular and irregular measures on groups and dyadic spaces</i> .....	173
Habib Salehi, <i>On interpolation of <math>q</math>-variate stationary stochastic processes</i> .....	183
Michael Samuel Skaff, <i>Vector valued Orlicz spaces generalized <math>N</math>-functions. I</i> .....	193
A. J. Ward, <i>On <math>H</math>-equivalence of uniformities. II</i> .....	207
Thomas Paul Whaley, <i>Algebras satisfying the descending chain condition for subalgebras</i> .....	217
G. K. White, <i>On subgroups of fixed index</i> .....	225
Martin Michael Zuckerman, <i>A unifying condition for implications among the axioms of choice for finite sets</i> .....	233