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NONOSCILLATORY SOLUTIONS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

LYNN HARRY ERBE

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We consider here a generalization of the equation

$$x'' + a(t)x^{2n+1} = 0$$

where $a(t)$ is a continuous non-negative function on $[0, +\infty)$ and $n \geq 0$ is an integer. Necessary and sufficient conditions are given for the existence of

- (1) a bounded nonoscillatory solution with prescribed limit at ∞ ;
- (2) a nonoscillatory solution whose derivative has a positive limit at ∞ .

Specifically, we are concerned with the asymptotic behavior of the solutions of the following second order nonlinear differential equation :

$$(1) \quad x'' + f(t, x)g(x') = 0 .$$

We shall assume the following conditions hold :

$$(A_0) \quad f(t, x), g(x'), \text{ and the partial derivative function } f_x(t, x) \text{ are all continuous for } t \geq 0, x' \geq 0, \text{ and } |x| < +\infty .$$

$$(A_1) \quad f(t, 0) = 0, t \geq 0 .$$

$$(A_2) \quad f_x(t, x) \geq 0 \text{ and is nondecreasing in } x \text{ for } t \geq 0 \text{ and } x \geq 0 .$$

$$(A_3) \quad g(x') > 0 \text{ for all } x' \geq 0 .$$

As a special case we have the equation

$$(2) \quad x'' + a(t)x^{2n+1} = 0, n \geq 0 ,$$

in which $a(t) \geq 0$ for $t \geq 0$ and $g(x') = 1$ for all x' . Oscillatory and nonoscillatory properties of (2) for the case $n \geq 1$ were investigated by Atkinson in [1], Moore and Nehari in [5], and Utz in [9]. Generalizations of equation (2) have been considered by Waltman in [7] and [8], Nehari in [6], Wong in [10], and Macki and Wong in [4].

We shall study equation (1) by considering the equation

$$(3) \quad x'' + f_x(t, \alpha)x = 0 ,$$

where α is some real constant depending on solutions of (1). To do this we shall need to establish several lemmas concerning the equation

$$(4) \quad x'' + p(t)x = 0,$$

where $p(t)$ is continuous and satisfies $p(t) \geq 0$ for $t \geq 0$.

LEMMA 1.1. *Let $[a, b]$ be a compact interval of the reals and suppose there exists a $\beta(t) \in C^{(2)} [a, b]$ satisfying*

$$\beta(t) > 0, \quad \beta''(t) + p(t)\beta(t) \leq 0, \quad t \in [a, b].$$

Then $[a, b]$ is an interval of disconjugacy for equation (4). That is, no nontrivial solution of (4) has more than one zero on $[a, b]$.

Proof. If the conclusion is false, then there is a solution $y(t)$ of (4) satisfying $y(t_1) = y(t_2) = 0$ and $y(t) > 0$ on (t_1, t_2) , where $a \leq t_1 < t_2 \leq b$. It follows that there is a $k > 0$ such that $ky(t) \leq \beta(t)$ on $[t_1, t_2]$ and $ky(t_0) = \beta(t_0)$ for some $t_1 < t_0 < t_2$. Therefore, $ky'(t_0) = \beta'(t_0)$ and for $t_0 \leq t \leq t_2$ we have

$$ky'(t) - \beta'(t) \geq \int_{t_0}^t -p(s)\{ky(s) - \beta(s)\}ds \geq 0.$$

Hence,

$$ky(t_2) - \beta(t_2) = \int_{t_0}^{t_2} (ky'(s) - \beta'(s))ds \geq 0,$$

which is a contradiction.

REMARK. If there exists an $\alpha(t) \in C^{(2)} [a, b]$ satisfying

$$\alpha(t) < 0, \quad \alpha''(t) + p(t)\alpha(t) \geq 0, \quad t \in [a, b],$$

then the conclusion of the lemma again holds. (Set $\beta(t) = -\alpha(t)$, $t \in [a, b]$.)

Lemma 1.1 is closely related to a theorem of Wintner (see Hartman [2], p. 362, Th. 7.2) and could be obtained directly by setting $z = \beta'/\beta$. Also, a function $\beta(t) \in C^{(2)} [a, b]$ satisfying $\beta''(t) + p(t)\beta(t) \leq 0$ on $[a, b]$ is just a special case of an upper solution, as defined by Jackson in [3] for general nonlinear second order differential equations. Likewise $\alpha(t) \in C^{(2)} [a, b]$ satisfying $\alpha''(t) + p(t)\alpha(t) \geq 0$ on $[a, b]$ is a special case of a lower solution.

LEMMA 1.2. *Let $\alpha(t), \beta(t) \in C^{(2)} [a, b]$ and satisfy $\alpha''(t) + p(t)\alpha(t) \geq 0$, $\beta''(t) + p(t)\beta(t) \leq 0$, and $0 < \alpha(t) \leq \beta(t)$ on $[a, b]$. Then for any c, d with $\alpha(a) \leq c \leq \beta(a)$, $\alpha(b) \leq d \leq \beta(b)$, there is a unique solution $z(t)$ of (4) satisfying $z(a) = c$, $z(b) = d$, and $\alpha(t) \leq z(t) \leq \beta(t)$ on $[a, b]$.*

Proof. By Lemma 1.1, $[a, b]$ is an interval of disconjugacy for equation (4) so that the BVP

$$x'' + p(t)x = 0, \quad x(a) = c, \quad x(b) = d$$

has a unique solution $z(t)$ (see for example [2], p. 351). Since $z(t)$ cannot have more than one zero on $[a, b]$ and since initial value problems for (4) have unique solutions, it follows that $z(t) > 0$ on $[a, b]$. If the conclusion of the lemma is false, then assume, to be specific, that $z(t_1) - \beta(t_1) = z(t_2) - \beta(t_2) = 0$ and $z(t) > \beta(t)$ on (t_1, t_2) , where $a \leq t_1 < t_2 \leq b$. As in Lemma 1.1, there is a $k > 0$, $k < 1$, such that $0 < kz(t) \leq \beta(t)$ on $[t_1, t_2]$, and $kz(t_0) = \beta(t_0)$, $kz'(t_0) = \beta'(t_0)$ for some $t_1 < t_0 < t_2$. Since $kz(t_2) < z(t_2) = \beta(t_2)$, this leads to a contradiction as in Lemma 1.1. Hence, $z(t) \leq \beta(t)$ on $[a, b]$. A similar argument shows that $z(t) \geq \alpha(t)$ on $[a, b]$ and this proves the lemma.

LEMMA 1.3. *Let $\alpha(t), \beta(t) \in C^{(2)} [a, +\infty)$ with $\alpha''(t) + p(t)\alpha(t) \geq 0$, $\beta''(t) + p(t)\beta(t) \leq 0$, and $0 < \alpha(t) \leq \beta(t)$ on $[a, +\infty)$. Then for any $\alpha(a) \leq c \leq \beta(a)$ there is a solution $y(t) \in C^{(2)} [a, +\infty)$ of (4) satisfying $y(a) = c$ and $\alpha(t) \leq y(t) \leq \beta(t)$ on $[a, +\infty)$.*

Proof. By Lemma 1.2 for each $n \geq 1$ there is a solution $y_n(t) \in C^{(2)} [a, a+n]$ of (4) satisfying $y_n(a) = c$ and $\alpha(t) \leq y_n(t) \leq \beta(t)$ on $[a, a+n]$. Therefore, for each $N \geq 1$ $|y_n(t)|$ and hence $|y_n''(t)|$ are uniformly bounded on $[a, a+N]$ for all $n = N$. Since $y_n'(t) = y_n'(a) + \int_a^t y_n''(t) dt$, the $|y_n'(t)|$ are likewise bounded on $[a, a+N]$, uniformly for $n \geq N$. Now consider the sequence $\{y_n(t)\}_{n=1}^\infty$. By the Ascoli-Arzelà Theorem there is a subsequence $\{y_{n_k}(t)\}_{n_k=1}^\infty$ converging to a solution $z_1(t)$ of (4) on $[a, a+1]$. Inductively, for each $k \geq 2$ we obtain a subsequence $\{y_{n_k}^k(t)\}_{n_k=1}^\infty$ of $\{y_{n_k}^{k-1}(t)\}_{n_k=1}^\infty$ which converges to a solution $z_k(t)$ of (4) on $[a, a+k]$. Therefore, the diagonal sequence $\{y_{n_k}^k(t)\}_{k=1}^\infty$ converges uniformly on each compact subinterval of $[a, +\infty)$. That is,

$$z(t) = \lim_{k \rightarrow \infty} y_{n_k}^k(t), \quad t \in [a, +\infty),$$

is the desired solution.

2. After these preliminary lemmas, we are now in a position to establish necessary and sufficient conditions for the existence of certain types of solutions of (1).

THEOREM 2.1. *Assume $A_0 - A_3$ hold and let $\alpha_0 > 0$. Then the following statements are equivalent:*

(a) *For each $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of (1) satisfying $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$.*

(b) $\int_0^\infty t f_y(t, \alpha) dt < +\infty$ for $0 < \alpha < \alpha_0$.

Proof. (a) implies (b): Assume $\int_0^\infty t f_y(t, \alpha_1) dt = +\infty$ for some $0 < \alpha_1 < \alpha_0$ and let $\alpha_1 < \beta < \alpha_0$. Let $u_\beta(t)$ be the corresponding solution of (1) with $\lim_{t \rightarrow \infty} u_\beta(t) = \beta$. Let $\delta > 0$ be such that $\alpha_1 + \delta < \beta$ and let $T \geq 0$ be such that $t \geq T$ implies $u_\beta(t) \geq \alpha_1 + \delta$. Then for $t \geq T$

$$u_\beta'' = -f(t, u_\beta)g(u_\beta) \leq 0$$

so that u_β' decreases to a limit, and this limit clearly must be zero. Therefore, $u_\beta(t) \leq \beta$ for $t \geq T$ so that applying the Mean Value Theorem we get

$$\begin{aligned} f_y(t, \alpha_1) &\leq \frac{f(t, u_\beta(t)) - f(t, \alpha_1)}{u_\beta(t) - \alpha_1} \leq \frac{f(t, u_\beta(t))}{u_\beta(t) - \alpha_1} \\ &\leq \frac{u_\beta(t)}{u_\beta(t) - \alpha_1} \frac{f(t, u_\beta(t))}{u_\beta(t)} \leq \frac{\beta}{\delta} \frac{f(t, u_\beta(t))}{u_\beta(t)}, \end{aligned}$$

for $t \geq T$. Since $\lim_{t \rightarrow \infty} u_\beta'(t) = 0$, there is a $T_1 \geq T$ such that $t \geq T_1$ implies $g(u_\beta'(t)) \geq g(0)/2 > 0$. Hence, for $t \geq T_1$ we have

$$u_\beta''(t) = -f(t, u_\beta(t))g(u_\beta'(t)) \leq -k f_y(t, \alpha_1) u_\beta(t),$$

where $k = g(0)(\delta/2\beta)$. Also, $\alpha_1'' = 0 \geq -k f_y(t, \alpha_1) \alpha_1$. Therefore, by Lemma 1.3 there is a solution $z(t)$ of the equation

$$(5) \quad x'' + k f_y(t, \alpha_1) x = 0$$

satisfying $\alpha_1 \leq z(t) \leq u_\beta(t)$ on $[T_1, +\infty)$. Let $w(t) = z(t) \int_{T_1}^t ds/(z(s))^2$ for $t \geq T_1$. Then $w(t)$ is a solution of (5). Since $z''(t) \leq 0$ for $t \geq T_1$, we see that

$$w''(t) = z''(t) \int_{T_1}^t ds/(z(s))^2 \leq 0$$

for $t \geq T_1$ and hence $w'(t)$ decreases to a finite nonnegative limit. In fact, we have

$$w'(t) = 1/z(t) + z'(t) \int_{T_1}^t ds/(z(s))^2 \geq 1/z(t) \geq 1/\beta$$

for $t \geq T_1$. Hence, for sufficiently large t , say $t \geq T_0 \geq T_1$, we have $w(t) \geq t/2\beta$. Therefore, for $t \geq T_0$ we have

$$\begin{aligned} w'(t) - w'(T_0) &= -k \int_{T_0}^t f_y(s, \alpha_1) w(s) ds \\ &\leq (-k/2\beta) \int_{T_0}^t s f_y(s, \alpha_1) ds \leq 0. \end{aligned}$$

Therefore,

$$w'(T_0) \geq w'(t) + (k/2\beta) \int_{T_0}^t s f_y(s, \alpha_1) ds$$

for $t \geq T_0$, so that

$$\int_{T_0}^{\infty} s f_y(s, \alpha_1) ds < +\infty,$$

which is the desired contradiction.

Conversely, let $0 < \alpha < \alpha_0$ be given and let

$$M = \max \{g(x') : 0 \leq x' \leq \alpha\}.$$

Let $T \geq 0$ be such that

$$\int_T^{\infty} (s - T) f_y(s, \alpha) ds < 1/M \text{ and } \int_T^{\infty} f_y(s, \alpha) ds < 1/M.$$

We shall now define a sequence of functions on $[T, +\infty)$ in the following manner:

Let $y_0(t) = \alpha$, $t \geq T$. Now for $t \geq T$

$$0 \leq \int_t^{\infty} (s - t) f(s, \alpha) g(0) ds \leq \alpha \int_t^{\infty} (s - t) f_y(s, \alpha) g(0) ds \leq \alpha,$$

so that defining $y_1(t) = \alpha - \int_t^{\infty} (s - t) f(s, \alpha) g(0) ds$, $t \geq T$, we have $0 \leq y_1(t) \leq \alpha$. Differentiating $y_1(t)$ we have

$$0 \leq y_1'(t) = \int_t^{\infty} f(s, \alpha) g(0) ds \leq M\alpha \int_t^{\infty} f_y(s, \alpha) ds < \alpha.$$

Proceeding inductively, we define for all $k \geq 1$

$$y_{k+1}(t) = \alpha - \int_t^{\infty} (s - t) f(s, y_k(s)) g(y_k'(s)) ds, \quad t \geq T,$$

and obtain $0 \leq y_k(t)$, $y_k'(t) \leq \alpha$ for all $k \geq 1$. It follows that the sequences $y_k(t)$, $y_k'(t)$, and $y_k''(t)$ are uniformly bounded on $[T, T + n]$ for all $n \geq 1$. The Ascoli-Arzelà Theorem and a diagonalization argument yields a subsequence which converges, uniformly on compact subsets of $[T, +\infty)$, to a solution $u_\alpha(t)$ of (1). Obviously, $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$. This completes the proof of the theorem.

REMARK. If $f(t, x) = -f(t, -x)$ and $g(x') > 0$ and is continuous for $|x'| < +\infty$, then we see that $\int_t^{\infty} t f_y(t, \alpha) dt < +\infty$ for $0 < |\alpha| < \alpha_0$ if and only if for each $0 < |\alpha| < \alpha_0$ there is a solution $u_\alpha(t)$ of (1) with $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$.

COROLLARY 2.2. $\int_0^\infty t f_y(t, \alpha) dt < +\infty$ for all $\alpha > 0$ if and only if there is a solution $u_\alpha(t)$ of (1) with $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ for all $\alpha > 0$.

COROLLARY 2.3. If $f(t, x) = \sum_{i=0}^n a_i(t) x^{2i+1}$ where the $a_i(t)$ are continuous nonnegative functions for $t \geq 0$, then the following statements are equivalent:

(a) There is a solution $u_\alpha(t)$ of (1) with $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$ for all $\alpha \neq 0$.

(b) $\sum_{i=0}^n \int_0^\infty t a_i(t) dt < +\infty$.

As examples of equations to which Theorem 2.1 applies but which do not belong to any of the classes of equations considered in references [1], [4] through [8], we have

$$(6) \quad x'' + x(\exp(t(x - \alpha_0)))(1 + x') = 0$$

$$(7) \quad x'' + x(\exp(t(x^2 - \alpha_0^2) + cx'))(1 + (x')^2) = 0,$$

where c is an arbitrary real number. Then for $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of (6) with $\lim_{t \rightarrow \infty} u_\alpha(t) = \alpha$, and for $0 < |\alpha| < \alpha_0$ there is a solution $y_\alpha(t)$ of (7) with $\lim_{t \rightarrow \infty} y_\alpha(t) = \alpha$.

3. In [5] it is shown that equation (2) has solutions for which

$$\lim_{t \rightarrow \infty} \frac{y(t)}{t} = \alpha > 0$$

if and only if

$$\int_0^\infty t^{2n+1} a(t) dt < +\infty.$$

In this final section we will show that an analogous result is true for equation (1) provided $f(t, x)$ satisfies the following additional condition.

(A) There exist real numbers $c > 0$ and $\lambda > 0$ such that

$$\liminf_{x \rightarrow \infty} \frac{f(t, x)}{x f_x(t, cx)} \geq \lambda > 0, \text{ for all sufficiently large } t.$$

Note that in the case of equation (2) c and λ may be any positive real numbers with $\lambda c^{2n} \leq 1/(2n + 1)$. We first establish the following lemma.

LEMMA 3.1. Assume conditions $A_0 - A_3$ hold and let there exist a real number $\beta > 0$ with

$$\int_0^\infty t f_y(t, \beta t) dt < +\infty.$$

Then there exist solutions to (1), say $y(t)$, such that $\lim_{t \rightarrow \infty} y(t)/t$ exists and is positive.

Proof. Let $T > 0$ be such that

$$\int_T^\infty t f_y(t, \beta t) dt < 1/2M,$$

where $M = \max \{g(x') : 0 \leq x' \leq \beta\}$. We define a solution of (1) by

$$u(T) = 0, \quad u'(T) = \beta,$$

and we assert that the solution satisfies $u'(t) \geq \beta/2$ for $t \geq T$. Assume, on the contrary, that there is a $\delta > 0$, $\beta/2 > \delta > 0$, and a $t_1 > T$ with $u'(t_1) = \delta$ and $u(t) > 0$ on $(T, t_1]$. Then for $T \leq t \leq t_1$ we have

$$(8) \quad u'(T) = u'(t) + \int_T^t f(s, u(s))g(u'(s))ds.$$

Since $u''(t) \leq 0$ on $(T, t_1]$ and since $u(t)$ is concave it follows that

$$\begin{aligned} u'(t) &\leq \beta \quad \text{on } (T, t_1) \quad \text{and} \\ u(t) &\leq \beta(t - T) \quad \text{on } (T, t_1). \end{aligned}$$

Applying the Mean Value Theorem in (8) we have

$$\begin{aligned} \beta &= u'(T) < u'(t) + M\beta \int_T^t s f_y(s, \beta(s - T))ds \\ &\leq u'(t) + M\beta \int_T^t s f_y(s, \beta s)ds < u'(t) + \beta/2. \end{aligned}$$

Hence, $u'(t_1) > \beta/2$, a contradiction. Therefore, $u'(t) \geq \beta/2$ on $[T, +\infty)$ and hence $\lim_{t \rightarrow \infty} u'(t)$ exists and is positive which implies that $\lim_{t \rightarrow \infty} u(t)/t$ exists and is positive.

THEOREM 3.2. *Assume conditions $(A_0) - (A_4)$ hold. Then (1) has solutions, say $y(t)$, such that $\lim_{t \rightarrow \infty} y(t)/t$ exists and is positive if and only if*

$$\int^\infty t f_y(t, \beta t) dt < +\infty \quad \text{for some } \beta_+ > 0.$$

Proof. Let $\alpha > 0$ and let $y(t)$ be a solution of (1) with

$$\lim_{t \rightarrow \infty} \frac{y(t)}{t} = \alpha.$$

Let $T \geq 0$ be such that $t \geq T$ implies $y(t) \geq \alpha t/2$. Let

$$m_0 = \min \{g(x') : 0 \leq x' \leq y'(T)\}.$$

By condition (A_1) there is a $T_1 \geq T$ such that $t \geq T_1$ implies

$$f(t, y(t)) \geq \lambda y(t) f_y(t, c\alpha t/2) \geq (kt) f_y(t, c\alpha t/2),$$

where $k = \lambda\alpha/2$. Since $0 < y'(t) \leq y'(T)$ for $t \geq T$ we have

$$f(t, y(t))g(y'(t)) \geq (m_0 kt) f_y(t, c\alpha t/2), \quad t \geq T_1.$$

Therefore,

$$\begin{aligned} y'(T_1) &= y'(t) + \int_{T_1}^t f(s, y(s))g(y'(s))ds \\ &\geq y'(t) + \int_{T_1}^t (m_0 ks) f_y(s, c\alpha s/2)ds. \end{aligned}$$

Since $\lim_{t \rightarrow \infty} y'(t) \geq 0$, this implies that

$$\int_{T_1}^{\infty} s f_y(s, c\alpha s/2)ds < +\infty,$$

and this proves the theorem.

As a simple example of an equation to which the previous theorem applies but which is not considered in references [1], [4] through [8], we have

$$(9) \quad x'' + x^2 (\exp(x - \beta t))(1 + x') = 0,$$

where $\beta > 0$. Condition (A_1) holds for any $0 < c < 1$ and any $\lambda > 0$.

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