

# Pacific Journal of Mathematics

**GAP SERIES AND AN EXAMPLE TO MALLIAVIN'S THEOREM**

ROBERT P. KAUFMAN

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**O. Malliavin's celebrated theorem of spectral nonsynthesis is based on a real function  $f$  of class  $A$**

$$f(t) = \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt ,$$

$$\sum |a_n| + \sum |b_n| < \infty ,$$

for which  $\int_{-\infty}^{\infty} |u| \|e^{iuf}\|_{\infty} du < \infty$ .

**Here and in general  $\|g\|_{\infty} \equiv \sup_n |\hat{g}(n)|$ . This note presents a method for constructing a function  $f$ , based on a gap property and a method of estimation of Kahane.**

Let  $0 < n_1 < n_2 < \dots < n_k < \dots$  be a sequence of integers with the property:

Whenever  $\varepsilon_k = 0, \pm 1$ , and  $\varepsilon_1 n_1 + \dots + \varepsilon_N n_N = 0$ , then  $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_N = 0$ .

Let  $\omega_1, \omega_2, \dots, \omega_k, \dots$  be independent random variables defined upon a probability space  $\Omega$ , distributed uniformly upon  $[0, 2\pi]$ . For a number  $0 < b < 1$  set

$$f(t) = \sum_{k=1}^{\infty} b^k \cos (n_k t + \omega_k) .$$

Then, for each integer  $M \geq 1$  there is a  $b = b(M) < 1$  such that

$$(1) \quad \int_{-\infty}^{\infty} |u|^M \|e^{iuf}\|_{\infty} du < \infty \quad \text{for almost all } \omega \text{ in } \Omega .$$

REMARKS. Choosing  $n_k = 2^k$ , we obtain a function  $f$  of class  $\text{Lip}(-\log b/\log 2)$ , and this shows that  $b(M)$  must converge to 1 as  $M \rightarrow \infty$ . For if the integral in (1) is finite, there is a number  $\xi$  such that  $(f - \xi)^M$  does not admit synthesis, and it must be false that

$$|f(t) - \xi|^{2M} = O(d(t, f^{-1}(\xi))) ,$$

[3, pp. 116, 122]. But then  $f \notin \text{Lip}(2^{-1/M})$ . Functions  $f$  with the Lipschitz condition were first produced in [1], and an explicit example—that is, nonprobabilistic—given in [2].

1. Let  $0 < r < 1$ ,  $0 < \varepsilon$ ,  $0 < \eta < (1 - r) \log 5 - \log 4$ . Define  $B_N(s, t)$  for  $0 < s, t < 2\pi(N = 1, 2, 3, \dots)$  to be the number of integers  $k$  defined by

$$1 \leq k \leq N, \quad |\cos n_k s - \cos n_k t| \geq \varepsilon .$$

LEMMA. *If  $\varepsilon > 0$  is small enough, the Lebesgue measure*

$$m\{B_N(s, t) \leq rN\} = O(e^{-\eta M}), \quad \text{as } N \rightarrow \infty .$$

*Proof.* Set

$$\xi_k(s, t) = 5 - (\cos n_k s - \cos n_k t)^2$$

or

$$\xi_k = 4 - \frac{1}{2} \cos 2n_k s + 2 \cos n_k s \cos n_k t - \frac{1}{2} \cos 2n_k t .$$

The mean of the product  $\xi_1 \cdots \xi_N$  is  $4^N$ . For the product is a sum of terms

$$c \Pi' \cos 2n_k s \Pi'' \cos n_k s \cos n_k t \Pi''' \cos 2n_k t ,$$

where the symbols  $\Pi'$ , etc., refer to products over mutually disjoint subsets of  $\{1, 2, \dots, N\}$ . If such a sum has mean  $\neq 0$ , it is trivial, for there are integers  $\varepsilon_k = \pm 1$ ,  $\delta_k = \pm 1$ , defined for every exponent  $n_k$  present, such that  $2\Sigma'\varepsilon_k n_k + \Sigma''\varepsilon_k n_k = \Sigma''\delta_k n_k + 2\Sigma'''\delta_k n_k = 0$ . But  $\Sigma'\varepsilon_k n_k + \frac{1}{2}\Sigma''(\varepsilon_k + \delta_k)n_k + \Sigma'''\delta_k n_k = 0$ , where  $\frac{1}{2}(\varepsilon_k - \delta_k) = 0, \pm 1$ . Thus  $\Pi'$  and  $\Pi'''$  must be trivial, and so finally  $\Pi''$  is trivial.

Now

$$\{B_N \leq rN\} \subseteq \{\xi_1 \cdots \xi_N \geq (5 - \varepsilon^2)^{(1-r)N}\} ,$$

so

$$m\{B_N \leq rN\} \leq 4\pi^2 [4/(5 - \varepsilon^2)^{1-r}]^N ,$$

and we need only choose  $\varepsilon > 0$  so that  $\eta < (1 - r) \log(5 - \varepsilon^2) - \log 4$ . We now choose  $\varepsilon > 0, \eta > 0, 1 > r > 0$ , once and for all.

2. Following [1] we observe that for  $g$  in  $L^2$

$$g(t) = \sum_{-\infty}^{\infty} c_n e^{int}$$

$$(g * g)(t) = (2\pi)^{-1} \int g(t - s)g(s)ds = \sum_{-\infty}^{\infty} c_n^2 e^{int}$$

$$\|g * g\|_2^2 = (2\pi)^{-1} \iiint g(t - s)g(s)g(\overline{t - p})g(\overline{p})dsdtdp = \sum_{-\infty}^{\infty} |c_n|^4 \geq \|g\|_{\infty}^4 .$$

Set

$$\begin{aligned} P(x, y, z, \omega) &= \cos(x - y + \omega) + \cos(y + \omega) - \cos(x - z + \omega) - \cos(z + \omega) . \end{aligned}$$

For fixed  $x, y, z, P$  is a trigonometric monomial in  $\omega$ , say  $\tau \sin(\omega + c)$ , and  $\tau$  can be estimated by setting

$$z' = z - \frac{1}{2}x, \quad y' = y - \frac{1}{2}x.$$

We find that  $\tau^2 = 4|\cos z' - \cos y'|^2$ . Now

$$\begin{aligned} & \exp iu[f(t - s) + f(s) - f(t - p) - f(p)] \\ &= \exp iu \sum_{k=1}^{\infty} b^k P(n_k t, n_k s, n_k p, \omega_k). \end{aligned}$$

To obtain an upper bound for the expectation of  $\|e^{iu f}\|_{\infty}^4$  we integrate this formula, first with respect to  $\omega_1, \omega_2, \dots$  and then with respect to  $s, p, t$ . Note the estimation

$$J_0(R) = (2\pi)^{-1} \int_0^{2\pi} e^{iR \sin \omega} d\omega \leq C(1 + |R|)^{-1/2}, \quad -\infty < R < \infty.$$

$$\begin{aligned} & (2\pi)^{-3} \iiint \prod_{k=1}^{\infty} |J_0(2ub_k \cdot |\cos n_k y' - \cos n_k z'|)| dx dy dz \\ & \leq (2\pi)^{-2} \iint \prod_1^{N(u)} |J_0(2ub^k \cdot |\cos n_k y - \cos n_k z'|)| dy dz. \end{aligned}$$

Here  $N(u)$  is the integral part of  $-\frac{1}{2} \log u / \log b$ . If  $B_{N(u)}(y, z) \geq rN(u)$  the product in the integral is at most  $(C'|u|^{-1/4})^{rN(u)}$ , a magnitude ultimately smaller than any assigned power of  $|u|^{-1}$ . The integral on the complement  $\{B_{N(u)} \leq rN(u)\}$  is  $O(e^{-\eta N(u)}) = O(|u|^{2^{-1/\eta} / \log b})$ . Choosing  $b$  close to 1, we can make this  $O(|u|^{-4M-6})$ . Then by Fubini's theorem

$$E\left(\int_{-\infty}^{\infty} |u|^{4M+4} \|e^{iu f}\|_{\infty}^4 du\right) = \int_{-\infty}^{\infty} |u|^{4M+4} E(\|e^{iu f}\|_{\infty}^4) du < \infty,$$

so  $\int_{-\infty}^{\infty} |u|^{4M+4} \|e^{iu f}\|_{\infty}^4 du < \infty$  for almost all  $\omega$  in  $\Omega$ . Conclusion (1) is a consequence of Holder's inequality.

It is clear that if  $b^k$  is replaced by  $k^{-2}$  for example, the condition (1) is valid for any integer  $M$ .

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