

# Pacific Journal of Mathematics

**ON UNICITY OF CAPACITY FUNCTIONS**

AKIO OSADA

## ON UNICITY OF CAPACITY FUNCTIONS

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**Sario's capacity function of a closed subset  $\gamma$  of the ideal boundary is known to be unique if  $\gamma$  is of positive capacity. The present paper will determine the number of capacity functions of  $\gamma$  in terms of the Heins harmonic dimension when  $\gamma$  has zero capacity, under the assumption that  $\gamma$  is isolated. This includes the special case where  $\gamma$  is the ideal boundary.**

1. Capacity functions. Denote by  $\beta$  the ideal boundary of an open Riemann surface  $R$  in the sense of Kerékjártó-Stoilow. We consider a fixed nonempty closed subset  $\gamma \subset \beta$  which is *isolated* from  $\delta = \beta - \gamma$ . Throughout this paper  $D$  will denote a fixed parametric disk about a fixed point  $\zeta \in R$  with a fixed local parameter  $z$  and the uniqueness is always referred to this fixed triple  $(\zeta, D, z)$ . Here we do not exclude the case where  $\gamma = \beta$ .

For a regular region  $\Omega \supset \bar{D}$  we denote by  $\gamma_\Omega$  the part of  $\partial\Omega$  which is "homologous" to  $\gamma$ . The remainder  $\delta_\Omega = \partial\Omega - \gamma_\Omega$  consists of a finite number of analytic Jordan curves  $\delta_{\Omega_j}$ . For a regular exhaustion  $\{R_n\}_{n=0}^\infty$  with  $R_0 \supset \bar{D}$  and nonempty  $\gamma_{R_0}$ , set  $\gamma_n = \gamma_{R_n}$  and  $\delta_{nj} = \delta_{R_{nj}}$ . Then there exists a unique function  $p_{r_n} \in H(R_n - \zeta)$  satisfying

$$(a) \quad p_{r_n}|_D = \log|z - \zeta| + h_n(z) \text{ with } h_n \in H(\bar{D}) \text{ and } h_n(\zeta) = 0,$$

$$(b) \quad p_{r_n}|_{\gamma_n} = k_n(\gamma) \text{ (const.) and } p_{r_n}|_{\delta_{nj}} = d_{nj} \text{ (const.) so that}$$

$$\int_{\delta_{nj}} *dp_{r_n} = 0, \text{ which is called a capacity function of } \gamma_n \text{ (Sario [6]).}$$

It is known that  $k_n(\gamma)$  increases with  $n$  and the limit  $k(\gamma)$  is independent of the choice of  $\{R_n\}_{n=0}^\infty$ . We call  $e^{-k(\gamma)}$  the capacity of  $\gamma$  and denote it by  $\text{cap } \gamma$ . When  $\text{cap } \gamma > 0$ ,  $p_{r_n}$  converges to a function  $p_\gamma$ , which is independent of the choice of the exhaustion (Sario [6]). Even when  $\text{cap } \gamma = 0$ , we can also choose a subsequence of  $\{p_{r_n}\}$  which converges to a function  $p_\gamma$ . Such functions  $p_\gamma$  will be called capacity functions of  $\gamma$  (Sario [6]). As mentioned above there exists only one capacity function when  $\text{cap } \gamma > 0$ .

It is the purpose of this paper to determine the number of capacity functions  $p_\gamma$  when  $\text{cap } \gamma = 0$ .

2. The harmonic dimension of  $\gamma$ . Let  $R, \beta, \gamma$  and  $\delta$  be as in 1. Furthermore we suppose that  $\gamma$  is of zero capacity. For a regular region  $\Omega \supset \bar{D}$  we denote by  $V_{\Omega_i}$  components of  $R - \bar{\Omega}$  whose derivations are contained in  $\gamma$  and by  $W_{\Omega_j}$  the remaining components. Here an ideal boundary component will be called a derivation of  $V_{\Omega_i}$  when it is contained in the closure of  $V_{\Omega_i}$  in the compactification of  $R$ . Here-

after we always choose  $\Omega$  so large as to make the derivations of  $W_\alpha = \bigcup_j W_{\alpha_j}$  contain in  $\delta$ . Therefore  $W_\alpha$  is always a neighborhood of all of  $\delta$ .

We consider the normal operator  $L_1^{(\alpha)}$  with respect to  $R - \bar{\Omega}$  associated with the partition  $P = \gamma + \sum_j \delta_j$  of  $\beta$  where  $\delta_j$  is a component of  $\delta$  (Ahlfors-Sario [1]).

Let  $q$  be a harmonic function in  $R - \zeta$ . Then  $q$  will be called of  $L_1$ -type at  $\delta$  when  $q = L_1^{(\alpha)}q$  in  $W_\alpha$  for an admissible  $\Omega$ . It is easy to see that this property depends only on  $\delta$ , i.e., if  $q = L_1^{(\alpha)}q$  in  $W_\alpha$ , then  $q = L_1^{(\alpha')}q$  in  $W_{\alpha'}$  for every admissible  $\Omega'$ .

We denote by  $HP_0(V_\alpha)$  the family of functions  $u$  such that  $u$  is a positive harmonic function in  $V_\alpha = \bigcup_i V_{\alpha_i}$  with boundary values zero at  $\gamma_\alpha = \partial V_\alpha$ . We may extend  $u$  to be identically zero in  $W_\alpha$ . Moreover we consider the following two families of functions. The first family  $N_\alpha$  consists of  $u \in HP_0(V_\alpha)$  such that  $\int_{\gamma_\alpha} *du = 2\pi$  where  $\gamma_\alpha$  is positively oriented with respect to  $\Omega$ . The second family is the family  $F$  of  $q \in H(R - \zeta)$  having the following properties:

- (c)  $q|_D = \log|z - \zeta| + h(z)$  with  $h \in H(\bar{D})$  and  $h(\zeta) = 0$ ,
- (d)  $q$  is of  $L_1$ -type at  $\delta$ ,
- (e)  $q$  is bounded from below near  $\gamma$ .

In addition to the obvious fact that  $N_\alpha$  and  $F$  are convex, they are related to each other as follows.

LEMMA. *There exists a bijective map  $T$  of  $N_\alpha$  onto  $F$  satisfying*

- (f)  $T(\lambda u + (1 - \lambda)v) = \lambda Tu + (1 - \lambda)Tv$  for  $u, v \in N_\alpha$ ,  $0 < \lambda < 1$ ,
- (g)  $Tu - u$  is bounded in  $V_\alpha$ .

For the proof let  $u \in N_\alpha$  and denote by  $L$  the direct sum of  $L_1^{(\alpha)}$  and the Dirichlet operator with respect to  $D$  (Sario [5]). Take the singularity function  $s_u$  on  $(R - \bar{\Omega}) \cup (D - \zeta)$  defined by  $s_u = u$  in  $R - \bar{\Omega}$  and  $s_u = \log|z - \zeta|$  in  $D - \zeta$ . Since the total flux of  $s_u$  is zero, the equation  $p - s_u = L(p - s_u)$  has a unique solution  $p_u$  on  $R$ , up to an additive constant. Normalize  $p_u$  so as to satisfy (c) and set  $Tu = p_u$ . Obviously  $Tu \in F$ . Since  $\gamma$  is of zero capacity,  $T$  is clearly injective. The property in (f) and (g) follows easily from the definition of  $T$ .

To see the surjectivity let  $q \in F$ . We denote by  $Bq$  the bounded harmonic function in  $V_\alpha$  with the boundary values  $q|_{\gamma_\alpha}$  at  $\gamma_\alpha$ . Set  $u = q - Bq$  in  $V_\alpha$  and  $u = 0$  in  $W_\alpha$ . Since  $q$  is of  $L_1$ -type at  $\delta$  and bounded from below near  $\gamma$ ,  $u \in N_\alpha$ . Therefore we have only to show that  $q - s_u = L(q - s_u)$  in  $(R - \bar{\Omega}) \cup (D - \zeta)$ . By the definition of  $u$ ,  $q - u = Bq$  in  $V_\alpha$  and  $L_1^{(\alpha)}(q - u) = L_1^{(\alpha)}q$  in  $V_\alpha$ . Furthermore  $Bq - L_1^{(\alpha)}q$  is bounded in  $V_\alpha$  and vanishes on  $\gamma_\alpha$ . Hence  $Bq = L_1^{(\alpha)}q$

in  $V_\rho$ . On the other hand,  $L_1^{(\rho)}(q - u) = L_1^{(\rho)}q$  in  $W_\rho$ . Consequently  $q - u = L(q - u)$  also in  $W_\rho$ . Finally it is obvious that the same equality holds in  $D - \zeta$ .

3. We denote by  $M_\rho$  the set of all minimal function in  $HP_0(V_\rho)$  normalized as  $\int_{r_\rho}^* du = 2\pi$ . Lemma 2 guarantees that the cardinal number of  $M_\rho$  is independent of the choice  $\rho$ . Extending Heins' definition (Heins [3]), we call it the harmonic dimension of  $\gamma$ , which we shall denote by  $d_\gamma$ .

4. The number of capacity functions. We are now able to state our main result:

**THEOREM.** *Suppose that  $\gamma$  is an isolated closed subset of zero capacity in the ideal boundary of  $R$ . If the harmonic dimension of  $\gamma$  is 1, then the capacity function of  $\gamma$  is unique. If the harmonic dimension of  $\gamma$  is greater than 1, there are a continuum of capacity functions of  $\gamma$ .*

Denote by  $C_\gamma$  the family of all capacity functions of  $\gamma$ , by  $c_\gamma$  the cardinal number of  $C_\gamma$  and also by  $\psi$  the cardinal number of the continuum. Then the statement of our theorem can also be summarized in a single formula as follows:

$$(1) \quad c_\gamma = 1 + (d_\gamma - 1)\psi .$$

5. Before entering the proof we need two lemmas, which will be used to show that  $C_\gamma = F$ . Let  $R_n, \gamma_n$  and  $\delta_{nj}$  be as in 1. Set  $V_{ni} = V_{R_{ni}}$  and  $W_{nj} = W_{R_{nj}}$  (see 2). Moreover put  $\Omega_{0n} = R - \bar{V}_0 - \bar{W}_n$  with  $V_0 = \bigcup_i V_{0i}$  and  $W_n = \bigcup_j W_{nj}$ .

**LEMMA.** *Let  $p \in F$ . Then there exists a sequence  $\{p_n\}_{n=0}^\infty$  with  $p_n \in H(\Omega_{0n} - \zeta)$  satisfying*

- (h)  $p_n | D = \log |z - \zeta| + h_n(z)$  with  $h_n \in H(\bar{D})$  and  $h_n(\zeta) = 0$ ,
- (i)  $p_n | \gamma_0 = p + k_n$  (const.) and  $p_n | \delta_{nj} = d_{ni}$  (const.) with

$$\int_{\delta_{nj}}^* dp_n = 0 ,$$

- (j)  $\{p_n\}$  converges uniformly to  $p$  on any compact  $K$  with

$$\bar{K} \subset \Omega_0 = R - \bar{V}_0 - \zeta .$$

For the proof construct  $p_n$  with (h) and (i) by the linear operator method of Sario [5]. Denote by  $D_\epsilon$  a parametric disk about  $\zeta$  with

radius  $\varepsilon$  and by  $\alpha_\varepsilon$  its circumference. We orient  $\alpha_\varepsilon$  and  $\gamma_0$  negatively with respect to  $\Omega_{0n} - \bar{D}_\varepsilon$  and write according to Ahlfors-Sario [1]:

$$A_\varepsilon(p) = \int_{\alpha_\varepsilon + \gamma_0} p^* dp, \quad B_n(p) = \int_{\delta_n} p^* dp, \quad A_\varepsilon(p, q) = \int_{\alpha_\varepsilon + \gamma_0} p^* dq$$

and

$$B_n(p, q) = \int_{\delta_n} p^* dq.$$

For  $m > n$  we denote by  $D_{n,\varepsilon}(p_m - p_n)$  and  $D_n(p_m - p_n)$  Dirichlet integrals of  $p_m - p_n$  taken over  $\Omega_{0n} - \bar{D}_\varepsilon$  and  $\Omega_{0n}$  respectively. Since  $B_n(p_n) = 0$ ,  $B_n(p_n, p_m) = 0$ ,

$$D_{n,\varepsilon}(p_m - p_n) = B_n(p_m) + 2A_\varepsilon(p_n, p_m) - A_\varepsilon(p_n) - A_\varepsilon(p_m).$$

Observing that  $B_n(p_m) < 0$  and letting  $\varepsilon \rightarrow 0$ ,

$$(2) \quad D_n(p_m - p_n) \leq a_m - a_n \quad \text{where} \quad a_j = \int_{\gamma_0} p^* dp_j + 2\pi k_j \quad (j = n, m).$$

Moreover we construct another sequence  $q_n \in H(\Omega_{0n} - \zeta)$  satisfying

(h')  $q_n | D = \log |z - \zeta| + h'_n(z)$  with  $h'_n \in H(\bar{D})$  and  $h'_n(\zeta) = 0$ ,

(i')  $q_n | \gamma_0 = p + k'_n$  (const.) and the normal derivative of  $q_n$  vanishes on  $\delta_n$ . By the same way as above we obtain

$$(3) \quad D_n(q_m - q_n) \leq b_m - b_n \quad \text{where} \quad b_j = \int_{\gamma_0} p^* dq_j + 2\pi k'_j \quad (j = n, m)$$

and

$$(4) \quad D_n(p_n - q_n) = b_n - a_n.$$

From (2), (3) and (4) we see  $a_n$  is increasing and  $b_n$  is decreasing as  $n$  increases and that  $a_n \leq b_n$ . Therefore  $\lim_n a_n$  and  $\lim_n b_n$  exist and are finite. In particular it follows from (2) that  $p_n$  converges uniformly to  $p$  on any compact  $K$  with  $\bar{K} \subset \Omega_0$ .

6. The following lemma is easy to see and plays an important role in the proof of our theorem.

LEMMA. *Let  $p \in F$ . Then there exist an exhaustion  $\{R_n\}_{n=0}^\infty$  and a sequence  $\{p_n\}_{n=0}^\infty$  with  $p_n \in H(R_n - \zeta)$  having the properties (h) of Lemma 5 and*

(k)  $p_n | \gamma_n = p + k_n$  (const.) and  $p_n | \delta_{nj} = d_{nj}$  (const.) with

$$\int_{\delta_{nj}} p_n^* dp_n = 0,$$

(1)  $\{p_n\}$  converges uniformly to  $p$  on any compact  $K$  in  $R - \zeta$ .

Since  $\gamma$  has zero capacity we can see that there exists an Evans potential  $e_0$  for  $\gamma$ , i.e., a function  $e_0 \in H(R - \zeta)$  satisfying the following conditions (Nakai [4]):

$$(m) \quad e_0 |D = \log |z - \zeta| + w(z) \text{ with } w \in H(\bar{D}) \text{ and } w(\zeta) = 0,$$

$$(n) \quad e_0 \text{ is of } L_1\text{-type at } \delta,$$

$$(o) \quad \lim_{z \rightarrow \gamma} e_0(z) = +\infty.$$

Needless to say  $e_0 \in F$ .

7. **Proof of theorem.** Consider  $p_\lambda = \lambda e_0 + (1 - \lambda)q$  with a fixed  $q \in F$  and  $0 < \lambda < 1$ . It is clear that  $\lim_{z \rightarrow \gamma} p_\lambda(z) = +\infty$  and  $p_\lambda \in F$ . Therefore by Lemma 6 we obtain

$$(5) \quad \{p_\lambda\}_{0 < \lambda < 1} \subset C_\gamma.$$

On the other hand, obviously

$$(6) \quad C_\gamma \subset F.$$

Moreover observe that  $\lambda \rightarrow p_\lambda$  is injective if  $e_0 \neq q$ .

By the approximation theorem of Heins [2], we can see at once that if  $d_\gamma = 1$ , so is the cardinal number of  $F$ . It is trivial that the converse is valid. Hence  $c_\gamma = 1$  if and only if  $d_\gamma = 1$ .

Suppose that  $d_\gamma \geq 2$ . Then there exists a  $q \in F$  with  $q \neq e_0$ . By the injectivity of  $\lambda \rightarrow p_\lambda$ ,  $\psi \leq c_\gamma$ . Conversely it follows from (6) that  $c_\gamma \leq$  the cardinal number of  $F$  which is not greater than  $\psi$ . Thus  $c_\gamma = \psi$ . In either case, since  $d_\gamma \leq \psi$ , we have  $c_\gamma = 1 + (d_\gamma - 1)\psi$ .

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