ON UNICITY OF CAPACITY FUNCTIONS

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Sario’s capacity function of a closed subset γ of the ideal boundary is known to be unique if γ is of positive capacity. The present paper will determine the number of capacity functions of γ in terms of the Heins harmonic dimension when γ has zero capacity, under the assumption that γ is isolated. This includes the special case where γ is the ideal boundary.

1. Capacity functions. Denote by β the ideal boundary of an open Riemann surface R in the sense of Kerekjártó-Stoïlov. We consider a fixed nonempty closed subset γ ⊆ β which is isolated from δ = β − γ. Throughout this paper D will denote a fixed parametric disk about a fixed point ζ ∈ R with a fixed local parameter z and the uniqueness is always referred to this fixed triple (ζ, D, z). Here we do not exclude the case where γ = β.

For a regular region Ω ⊃ D we denote by γo the part of ∂Ω which is “homologous” to γ. The remainder δo = ∂Ω − γo consists of a finite number of analytic Jordan curves δoj. For a regular exhaustion \{Rn\}n=0 with R0 ⊃ D and nonempty γR0, set γn = γRn and δnj = δRnj. Then there exists a unique function pn ∈ H(Rn − ζ) satisfying

(a) pn|D = log|z − ζ| + hn(z) with hn ∈ H(D) and hn(ζ) = 0,
(b) pn|γn = kn(γ) (const.) and pn|δnj = dnj (const.) so that \[ \int_{\gamma_n} *dp_n = 0, \]

which is called a capacity function of γn (Sario [6]).

It is known that kn(γ) increases with n and the limit k(γ) is independent of the choice of \{Rn\}n=0. We call e−k(γ) the capacity of γ and denote it by cap γ. When cap γ > 0, pn converges to a functions pr, which is independent of the choice of the exhaustion (Sario [6]). Even when cap γ = 0, we can also choose a subsequence of \{pn\} which converges to a function pr. Such functions pr will be called capacity functions of γ (Sario [6]). As mentioned above there exists only one capacity function when cap γ > 0.

It is the purpose of this paper to determine the number of capacity functions pr when cap γ = 0.

2. The harmonic dimension of γ. Let R, β, γ and δ be as in 1. Furthermore we suppose that γ is of zero capacity. For a regular region Ω ⊃ D we denote by Vαi components of R − Ω whose derivations are contained in γ and by Wαj the remaining components. Here an ideal boundary component will be called a derivation of Vαi when it is contained in the closure of Vαi in the compactification of R. Here-
after we always choose $\Omega$ so large as to make the derivations of $W_\alpha = \bigcup_j W_{\alpha_j}$ contain in $\delta$. Therefore $W_\alpha$ is always a neighborhood of all of $\delta$.

We consider the normal operator $L^{(\alpha)}_i$ with respect to $R - \bar{\Omega}$ associated with the partition $P' = \gamma + \sum_j \delta_j$ of $\beta$ where $\delta_j$ is a component of $\delta$ (Ahlfors-Sario [1]).

Let $q$ be a harmonic function in $R - \zeta$. Then $q$ will be called of $L_i$-type at $\delta$ when $q = L^{(\alpha)}_i q$ in $W_\alpha$ for an admissible $\Omega$. It is easy to see that this property depends only on $\delta$, i.e., if $q = L^{(\alpha)}_i q$ in $W_\alpha$, then $q = L^{(\alpha')}_i q$ in $W_\alpha$ for every admissible $\Omega'$.

We denote by $HP_\Omega(V_\Omega)$ the family of functions $u$ such that $u$ is a positive harmonic function in $V_\Omega = \bigcup_i V_{\alpha_i}$ with boundary values zero at $\gamma_u = \partial V_\alpha$. We may extend $u$ to be identically zero in $W_\alpha$.

Moreover we consider the following two families of functions. The first family $N_\alpha$ consists of $u \in HP_\Omega(V_\Omega)$ such that $\int_{\partial \Omega} \nu u = 2\pi$ where $\gamma_\alpha$ is positively oriented with respect to $\Omega$. The second family is the family $F$ of $q \in H(R - \zeta)$ having the following properties:

(c) $q \mid_D = \log |z - \zeta| + h(z)$ with $h \in H(D)$ and $h(\zeta) = 0$,
(d) $q$ is of $L_i$-type at $\delta$,
(e) $q$ is bounded from below near $\gamma$.

In addition to the obvious fact that $N_\alpha$ and $F$ are convex, they are related to each other as follows.

**Lemma.** There exists a bijective map $T$ of $N_\alpha$ onto $F$ satisfying

(f) $T(\lambda u + (1 - \lambda)v) = \lambda Tu + (1 - \lambda)Tv$ for $u, v \in N_\alpha$, $0 < \lambda < 1$,
(g) $Tu - u$ is bounded in $V_\alpha$.

For the proof let $u \in N_\alpha$ and denote by $L$ the direct sum of $L^{(\alpha)}_i$ and the Dirichlet operator with respect to $D$ (Sario [5]). Take the singularity function $s_u$ on $(R - \bar{\Omega}) \cup (D - \zeta)$ defined by $s_u = u$ in $R - \bar{\Omega}$ and $s_u = \log |z - \zeta|$ in $D - \zeta$. Since the total flux of $s_u$ is zero, the equation $p - s_u = L(p - s_u)$ has a unique solution $p_u$ on $R$, up to an additive constant. Normalize $p_u$ so as to satisfy (c) and set $Tu = p_u$. Obviously $Tu \in F$. Since $\gamma$ is of zero capacity, $T$ is clearly injective. The property in (f) and (g) follows easily from the definition of $T$.

To see the surjectivity let $q \in F$. We denote by $Bq$ the bounded harmonic function in $V_\alpha$ with the boundary values $q \mid \gamma_\alpha$ at $\gamma_\alpha$. Set $u = q - Bq$ in $V_\alpha$ and $u = 0$ in $W_\alpha$. Since $q$ is of $L_i$-type at $\delta$ and bounded from below near $\gamma$, $u \in N_\alpha$. Therefore we have only to show that $q - s_u = L(q - s_u)$ in $(R - \bar{\Omega}) \cup (D - \zeta)$. By the definition of $u$, $q - u = Bq$ in $V_\alpha$ and $L^{(\alpha)}_i(q - u) = L^{(\alpha)}_i Bq$ in $V_\alpha$. Furthermore $Bq - L^{(\alpha)}_i q$ is bounded in $V_\alpha$ and vanishes on $\gamma_\alpha$. Hence $Bq = L^{(\alpha)}_i q$
in $V_\alpha$. On the other hand, $L_i^{(q - u)} = L_i^{(q)}$ in $W_\alpha$. Consequently $q - u = L(q - u)$ also in $W_\alpha$. Finally it is obvious that the same equality holds in $D - \zeta$.

3. We denote by $M_\alpha$ the set of all minimal function in $HP_i(V_\alpha)$ normalized as $\int_{\partial V}^* du = 2\pi$. Lemma 2 guarantees that the cardinal number of $M_\alpha$ is independent of the choice $\Omega$. Extending Heins' definition (Heins [3]), we call it the harmonic dimension of $\gamma$, which we shall denote by $d_\gamma$.

4. The number of capacity functions. We are now able to state our main result:

**Theorem.** Suppose that $\gamma$ is an isolated closed subset of zero capacity in the ideal boundary of $R$. If the harmonic dimension of $\gamma$ is $1$, then the capacity function of $\gamma$ is unique. If the harmonic dimension of $\gamma$ is greater than $1$, there are a continuum of capacity functions of $\gamma$.

Denote by $C_\gamma$ the family of all capacity functions of $\gamma$, by $c_\gamma$ the cardinal number of $C_\gamma$ and also by $\psi$ the cardinal number of the continuum. Then the statement of our theorem can also be summarized in a single formula as follows:

$$c_\gamma = 1 + (d_\gamma - 1)\psi.$$  

5. Before entering the proof we need two lemmas, which will be used to show that $C_\gamma = F$. Let $R_n$, $\gamma_n$ and $\delta_n$ be as in 1. Set $V_{ni} = V_{R_n i}$ and $W_{nj} = W_{R_n j}$ (see 2). Moreover put $\Omega_{\delta_n} = R - \overline{V}_0 - \overline{W}_n$ with $V_0 = \bigcup_i V_{oi}$ and $W_n = \bigcup_j W_{nj}$.

**Lemma.** Let $p \in F$. Then there exists a sequence $\{p_n\}_{n=0}^\infty$ with $p_n \in H(\Omega_{\delta_n} - \zeta)$ satisfying

- (h) $p_n | D = \log |z - \zeta| + h_n(z)$ with $h_n \in H(\overline{D})$ and $h_n(\zeta) = 0$,
- (i) $p_n | \gamma_0 = p + k_n$ (const.) and $p_n | \delta_n = d_{ni}$ (const.) with $\int_{\delta_n}^* dp_n = 0$,
- (j) $\{p_n\}$ converges uniformly to $p$ on any compact $K$ with $\overline{K} \subset \Omega_0 = R - \overline{V}_0 - \zeta$.

For the proof construct $p_n$ with (h) and (i) by the linear operator method of Sario [5]. Denote by $D_\varepsilon$ a parametric disk about $\zeta$ with
radius $\varepsilon$ and by $a_\varepsilon$ its circumference. We orient $a_\varepsilon$ and $\tau_0$ negatively with respect to $\Omega_{Q_0} - \bar{D}_\varepsilon$ and write according to Ahlfors-Sario [1]:

$$A_\varepsilon(p) = \int_{\omega + \varepsilon} p^* dp, \quad B_n(p) = \int_{\omega + \varepsilon} p^* dp, \quad A_\varepsilon(p, q) = \int_{\omega + \varepsilon} p^* dq$$

and

$$B_n(p, q) = \int_{\omega + \varepsilon} p^* dq.$$

For $m > n$ we denote by $D_{n, \varepsilon}(p_m - p_n)$ and $D_n(p_m - p_n)$ Dirichlet integrals of $p_m - p_n$ taken over $\Omega_{Q_0} - \bar{D}_\varepsilon$, and $\Omega_{Q_n}$ respectively. Since $B_n(p_n) = 0$, $B_n(p_n, p_m) = 0,$

$$D_{n, \varepsilon}(p_m - p_n) = B_n(p_m) + 2A_\varepsilon(p_n, p_m) - A_\varepsilon(p_n) - A_\varepsilon(p_m).$$

Observing that $B_n(p_m) < 0$ and letting $\varepsilon \to 0$,

$$D_{n, \varepsilon}(p_m - p_n) \leq a_m - a_n$$

where $a_j = \int_{\gamma_0} p^* dp_j + 2\pi k_j$ (j = $n$, $m$).

Moreover we construct another sequence $q_n \in H(\Omega_{Q_n} - \zeta)$ satisfying

(h') $q_n |D = \log |z - \zeta| + h'_n(z)$ with $h'_n \in H(\bar{D})$ and $h'_n(\zeta) = 0$,

(i') $q_n |\gamma_0 = p + k'_n$ (const.) and the normal derivative of $q_n$ vanishes on $\delta_n$. By the same way as above we obtain

(3) $D_n(q_m - q_n) \leq b_n - b_m$ where $b_j = \int_{\gamma_0} p^* dq_j + 2\pi k'_j$ (j = $n$, $m$) and

(4) $D_n(p_n - q_n) = b_n - a_n$.

From (2), (3) and (4) we see $a_n$ is increasing and $b_n$ is decreasing as $n$ increases and that $a_n \leq b_n$. Therefore $\lim_n a_n$ and $\lim_n b_n$ exist and are finite. In particular it follows from (2) that $p_n$ converges uniformly to $p$ on any compact $K$ with $\bar{K} \subset \Omega_{Q_0}$.

6. The following lemma is easy to see and plays an important role in the proof of our theorem.

**Lemma.** Let $p \in F$. Then there exist an exhaustion $\{R_n\}_{n=0}^{\infty}$ and a sequence $\{p_n\}_{n=0}^{\infty}$ with $p_n \in H(R_n - \zeta)$ having the properties (h) of Lemma 5 and

(k) $p_n |\gamma_n = p + k_n$ (const.) and $p_n |\delta_n = d_n$ (const.) with

$$\int_{\delta_n} p_n^* dp_n = 0,$$

(1) $\{p_n\}$ converges uniformly to $p$ on any compact $K$ in $R - \zeta$. 
Since \( \gamma \) has zero capacity we can see that there exists an Evans potential \( e_0 \) for \( \gamma \), i.e., a function \( e_0 \in H(R - \zeta) \) satisfying the following conditions (Nakai [4]):

\[
\begin{align*}
(m) & \quad e_0 | D = \log | z - \zeta | + w(z) \text{ with } w \in H(\bar{D}) \text{ and } w(\zeta) = 0, \\
(n) & \quad e_0 \text{ is of } L_\Gamma \text{-type at } \delta, \\
(o) & \quad \lim_{z \to \delta} e_0(z) = +\infty.
\end{align*}
\]

Needless to say \( e_0 \in F \).

7. Proof of theorem. Consider \( p_\lambda = \lambda e_0 + (1 - \lambda)q \) with a fixed \( q \in F \) and \( 0 < \lambda < 1 \). It is clear that \( \lim_{\lambda \to 0} p_\lambda(z) = +\infty \) and \( p_\lambda \in F \). Therefore by Lemma 6 we obtain

\[
(5) \quad \{ p_\lambda \}_0 < \lambda < 1 \subset C_\gamma.
\]

On the other hand, obviously

\[
(6) \quad C_\gamma \subset F.
\]

Moreover observe that \( \lambda \to p_\lambda \) is injective if \( e_0 \neq q \).

By the approximation theorem of Heins [2], we can see at once that if \( d_\gamma = 1 \), so is the cardinal number of \( F \). It is trivial that the converse is valid. Hence \( c_\gamma = 1 \) if and only if \( d_\gamma = 1 \).

Suppose that \( d_\gamma \geq 2 \). Then there exists a \( q \in F \) with \( q \neq e_0 \). By the injectivity of \( \lambda \to p_\lambda \), \( \psi \leq c_\gamma \). Conversely it follows from (6) that \( c_\gamma \leq \) the cardinal number of \( F \) which is not greater than \( \psi \). Thus \( c_\gamma = \psi \). In either case, since \( d_\gamma \leq \psi \), we have \( c_\gamma = 1 + (d_\gamma - 1)\psi \).

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**References**


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**NAGOYA UNIVERSITY**
Patrick Robert Ahern, *On the geometry of the unit ball in the space of real annihilating measures* ........................................... 1
Kirby Alan Baker, *Equational classes of modular lattices* ................. 9
E. F. Beckenbach and Gerald Andrew Hutchison, *Meromorphic minimal surfaces* .......................................................... 17
Tae Ho Choe, *Intrinsic topologies in a topological lattice* .................. 49
John Bligh Conway, *A theorem on sequential convergence of measures and some applications* .................................................. 53
Roger Cuppens, *On the decomposition of infinitely divisible probability laws without normal factor* ........................................ 61
Lynn Harry Erbe, *Nonoscillatory solutions of second order nonlinear differential equations* .................................................... 77
Burton I. Fein, *The Schur index for projective representations of finite groups* ................................................................. 87
Stanley P. Gudder, *A note on proposition observables* ....................... 101
Kenneth Kapp, *On Croisot’s theory of decompositions* ...................... 105
Robert P. Kaufman, *Gap series and an example to Malliavin’s theorem* 117
E. J. McShane, Robert Breckenridge Warfield, Jr. and V. M. Warfield, *Invariant extensions of linear functionals, with applications to measures and stochastic processes* .................................. 121
Marvin Victor Mielke, *Rearrangement of spherical modifications* ........ 143
Akio Osada, *On unicity of capacity functions* .................................. 151
Donald Steven Passman, *Some 5/2 transitive permutation groups* .......... 157
Harold L. Peterson, Jr., *Regular and irregular measures on groups and dyadic spaces* ....................................................... 173
Habib Salehi, *On interpolation of q-variate stationary stochastic processes* ............................................................................ 183
Michael Samuel Skaff, *Vector valued Orlicz spaces generalized* *N*-functions. I ................................................................. 193
Thomas Paul Whaley, *Algebras satisfying the descending chain condition for subalgebras* ................................................. 217
G. K. White, *On subgroups of fixed index* ........................................ 225
Martin Michael Zuckerman, *A unifying condition for implications among the axioms of choice for finite sets* ....................... 233