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ON UNICITY OF CAPACITY FUNCTIONS

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Sario's capacity function of a closed subset γ of the ideal boundary is known to be unique if γ is of positive capacity. The present paper will determine the number of capacity functions of γ in terms of the Heins harmonic dimension when γ has zero capacity, under the assumption that γ is isolated. This includes the special case where γ is the ideal boundary.

1. **Capacity functions.** Denote by β the ideal boundary of an open Riemann surface R in the sense of Kerékjártó-Stoilow. We consider a fixed nonempty closed subset $\gamma \subset \beta$ which is *isolated* from $\delta = \beta - \gamma$. Throughout this paper D will denote a fixed parametric disk about a fixed point $\zeta \in R$ with a fixed local parameter z and the uniqueness is always referred to this fixed triple (ζ, D, z) . Here we do not exclude the case where $\gamma = \beta$.

For a regular region $\Omega \supset \bar{D}$ we denote by γ_Ω the part of $\partial\Omega$ which is "homologous" to γ . The remainder $\delta_\Omega = \partial\Omega - \gamma_\Omega$ consists of a finite number of analytic Jordan curves $\delta_{\Omega j}$. For a regular exhaustion $\{R_n\}_{n=0}^\infty$ with $R_0 \supset \bar{D}$ and nonempty γ_{R_0} , set $\gamma_n = \gamma_{R_n}$ and $\delta_{nj} = \delta_{R_n j}$. Then there exists a unique function $p_{r_n} \in H(R_n - \zeta)$ satisfying

(a) $p_{r_n} | D = \log |z - \zeta| + h_n(z)$ with $h_n \in H(\bar{D})$ and $h_n(\zeta) = 0$,

(b) $p_{r_n} | \gamma_n = k_n(\gamma)$ (const.) and $p_{r_n} | \delta_{nj} = d_{nj}$ (const.) so that

$$\int_{\delta_{nj}} *dp_{r_n} = 0, \text{ which is called a capacity function of } \gamma_n \text{ (Sario [6]).}$$

It is known that $k_n(\gamma)$ increases with n and the limit $k(\gamma)$ is independent of the choice of $\{R_n\}_{n=0}^\infty$. We call $e^{-k(\gamma)}$ the capacity of γ and denote it by $\text{cap } \gamma$. When $\text{cap } \gamma > 0$, p_{r_n} converges to a functions p_γ , which is independent of the choice of the exhaustion (Sario [6]). Even when $\text{cap } \gamma = 0$, we can also choose a subsequence of $\{p_{r_n}\}$ which converges to a function p_γ . Such functions p_γ will be called capacity functions of γ (Sario [6]). As mentioned above there exists only one capacity function when $\text{cap } \gamma > 0$.

It is the purpose of this paper to determine the number of capacity functions p_γ when $\text{cap } \gamma = 0$.

2. **The harmonic dimension of γ .** Let R, β, γ and δ be as in 1. Furthermore we suppose that γ is of zero capacity. For a regular region $\Omega \supset \bar{D}$ we denote by $V_{\Omega i}$ components of $R - \bar{\Omega}$ whose derivations are contained in γ and by $W_{\Omega j}$ the remaining components. Here an ideal boundary component will be called a derivation of $V_{\Omega i}$ when it is contained in the closure of $V_{\Omega i}$ in the compactification of R . Here-

after we always choose Ω so large as to make the derivations of $W_\rho = \bigcup_j W_{\rho_j}$ contain in δ . Therefore W_ρ is always a neighborhood of all of δ .

We consider the normal operator $L_1^{(\rho)}$ with respect to $R - \bar{\Omega}$ associated with the partition $P = \gamma + \sum_j \delta_j$ of β where δ_j is a component of δ (Ahlfors-Sario [1]).

Let q be a harmonic function in $R - \zeta$. Then q will be called of L_1 -type at δ when $q = L_1^{(\rho)}q$ in W_ρ for an admissible Ω . It is easy to see that this property depends only on δ , i.e., if $q = L_1^{(\rho)}q$ in W_ρ , then $q = L_1^{(\rho')}q$ in $W_{\rho'}$ for every admissible ρ' .

We denote by $HP_0(V_\rho)$ the family of functions u such that u is a positive harmonic function in $V_\rho = \bigcup_i V_{\rho_i}$ with boundary values zero at $\gamma_\rho = \partial V_\rho$. We may extend u to be identically zero in W_ρ . Moreover we consider the following two families of functions. The first family N_ρ consists of $u \in HP_0(V_\rho)$ such that $\int_{\gamma_\rho} *du = 2\pi$ where γ_ρ is positively oriented with respect to Ω . The second family is the family F of $q \in H(R - \zeta)$ having the following properties:

- (c) $q|_D = \log|z - \zeta| + h(z)$ with $h \in H(\bar{D})$ and $h(\zeta) = 0$,
- (d) q is of L_1 -type at δ ,
- (e) q is bounded from below near γ .

In addition to the obvious fact that N_ρ and F are convex, they are related to each other as follows.

LEMMA. *There exists a bijective map T of N_ρ onto F satisfying*

- (f) $T(\lambda u + (1 - \lambda)v) = \lambda Tu + (1 - \lambda)Tv$ for $u, v \in N_\rho$, $0 < \lambda < 1$,
- (g) $Tu - u$ is bounded in V_ρ .

For the proof let $u \in N_\rho$ and denote by L the direct sum of $L_1^{(\rho)}$ and the Dirichlet operator with respect to D (Sario [5]). Take the singularity function s_u on $(R - \bar{\Omega}) \cup (D - \zeta)$ defined by $s_u = u$ in $R - \bar{\Omega}$ and $s_u = \log|z - \zeta|$ in $D - \zeta$. Since the total flux of s_u is zero, the equation $p - s_u = L(p - s_u)$ has a unique solution p_u on R , up to an additive constant. Normalize p_u so as to satisfy (c) and set $Tu = p_u$. Obviously $Tu \in F$. Since γ is of zero capacity, T is clearly injective. The property in (f) and (g) follows easily from the definition of T .

To see the surjectivity let $q \in F$. We denote by Bq the bounded harmonic function in V_ρ with the boundary values $q|_{\gamma_\rho}$ at γ_ρ . Set $u = q - Bq$ in V_ρ and $u = 0$ in W_ρ . Since q is of L_1 -type at δ and bounded from below near γ , $u \in N_\rho$. Therefore we have only to show that $q - s_u = L(q - s_u)$ in $(R - \bar{\Omega}) \cup (D - \zeta)$. By the definition of u , $q - u = Bq$ in V_ρ and $L_1^{(\rho)}(q - u) = L_1^{(\rho)}q$ in V_ρ . Furthermore $Bq - L_1^{(\rho)}q$ is bounded in V_ρ and vanishes on γ_ρ . Hence $Bq = L_1^{(\rho)}q$

in V_ρ . On the other hand, $L_1^{(\rho)}(q - u) = L_1^{(\rho)}q$ in W_ρ . Consequently $q - u = L(q - u)$ also in W_ρ . Finally it is obvious that the same equality holds in $D - \zeta$.

3. We denote by M_ρ the set of all minimal function in $HP_0(V_\rho)$ normalized as $\int_{\gamma_\rho} *du = 2\pi$. Lemma 2 guarantees that the cardinal number of M_ρ is independent of the choice ρ . Extending Heins' definition (Heins [3]), we call it the harmonic dimension of γ , which we shall denote by d_γ .

4. The number of capacity functions. We are now able to state our main result:

THEOREM. *Suppose that γ is an isolated closed subset of zero capacity in the ideal boundary of R . If the harmonic dimension of γ is 1, then the capacity function of γ is unique. If the harmonic dimension of γ is greater than 1, there are a continuum of capacity functions of γ .*

Denote by C_γ the family of all capacity functions of γ , by c_γ the cardinal number of C_γ and also by ψ the cardinal number of the continuum. Then the statement of our theorem can also be summarized in a single formula as follows:

$$(1) \quad c_\gamma = 1 + (d_\gamma - 1)\psi .$$

5. Before entering the proof we need two lemmas, which will be used to show that $C_\gamma = F$. Let R_n, γ_n and δ_{nj} be as in 1. Set $V_{ni} = V_{R_{ni}}$ and $W_{nj} = W_{R_{nj}}$ (see 2). Moreover put $\Omega_n = R - \bar{V}_0 - \bar{W}_n$ with $V_0 = \bigcup_i V_{0i}$ and $W_n = \bigcup_j W_{nj}$.

LEMMA. *Let $p \in F$. Then there exists a sequence $\{p_n\}_{n=0}^\infty$ with $p_n \in H(\Omega_n - \zeta)$ satisfying*

- (h) $p_n | D = \log |z - \zeta| + h_n(z)$ with $h_n \in H(\bar{D})$ and $h_n(\zeta) = 0$,
- (i) $p_n | \gamma_0 = p + k_n$ (const.) and $p_n | \delta_{nj} = d_{ni}$ (const.) with

$$\int_{\delta_{nj}} *dp_n = 0 ,$$

- (j) $\{p_n\}$ converges uniformly to p on any compact K with

$$\bar{K} \subset \Omega_0 = R - \bar{V}_0 - \zeta .$$

For the proof construct p_n with (h) and (i) by the linear operator method of Sario [5]. Denote by D_ϵ a parametric disk about ζ with

radius ε and by α_ε its circumference. We orient α_ε and γ_0 negatively with respect to $\Omega_{0n} - \bar{D}_\varepsilon$ and write according to Ahlfors-Sario [1]:

$$A_\varepsilon(p) = \int_{\alpha_\varepsilon + \gamma_0} p^* dp, \quad B_n(p) = \int_{\delta_n} p^* dp, \quad A_\varepsilon(p, q) = \int_{\alpha_\varepsilon + \gamma_0} p^* dq$$

and

$$B_n(p, q) = \int_{\delta_n} p^* dq.$$

For $m > n$ we denote by $D_{n,\varepsilon}(p_m - p_n)$ and $D_n(p_m - p_n)$ Dirichlet integrals of $p_m - p_n$ taken over $\Omega_{0n} - \bar{D}_\varepsilon$ and Ω_{0n} respectively. Since $B_n(p_n) = 0$, $B_n(p_n, p_m) = 0$,

$$D_{n,\varepsilon}(p_m - p_n) = B_n(p_m) + 2A_\varepsilon(p_n, p_m) - A_\varepsilon(p_n) - A_\varepsilon(p_m).$$

Observing that $B_n(p_m) < 0$ and letting $\varepsilon \rightarrow 0$,

$$(2) \quad D_n(p_m - p_n) \leq a_m - a_n \quad \text{where } a_j = \int_{\gamma_0} p^* dp_j + 2\pi k_j \quad (j = n, m).$$

Moreover we construct another sequence $q_n \in H(\Omega_{0n} - \zeta)$ satisfying

$$(h') \quad q_n | D = \log |z - \zeta| + h'_n(z) \quad \text{with } h'_n \in H(\bar{D}) \text{ and } h'_n(\zeta) = 0,$$

(i') $q_n | \gamma_0 = p + k'_n$ (const.) and the normal derivative of q_n vanishes on δ_n . By the same way as above we obtain

$$(3) \quad D_n(q_m - q_n) \leq b_n - b_m \quad \text{where } b_j = \int_{\gamma_0} p^* dq_j + 2\pi k'_j \quad (j = n, m)$$

and

$$(4) \quad D_n(p_n - q_n) = b_n - a_n.$$

From (2), (3) and (4) we see a_n is increasing and b_n is decreasing as n increases and that $a_n \leq b_n$. Therefore $\lim_n a_n$ and $\lim_n b_n$ exist and are finite. In particular it follows from (2) that p_n converges uniformly to p on any compact K with $\bar{K} \subset \Omega_0$.

6. The following lemma is easy to see and plays an important role in the proof of our theorem.

LEMMA. *Let $p \in F$. Then there exist an exhaustion $\{R_n\}_{n=0}^\infty$ and a sequence $\{p_n\}_{n=0}^\infty$ with $p_n \in H(R_n - \zeta)$ having the properties (h) of Lemma 5 and*

$$(k) \quad p_n | \gamma_n = p + k_n \text{ (const.) and } p_n | \delta_{nj} = d_{nj} \text{ (const.) with}$$

$$\int_{\delta_{nj}} p_n^* dp_n = 0,$$

$$(1) \quad \{p_n\} \text{ converges uniformly to } p \text{ on any compact } K \text{ in } R - \zeta.$$

Since γ has zero capacity we can see that there exists an Evans potential e_0 for γ , i.e., a function $e_0 \in H(R - \zeta)$ satisfying the following conditions (Nakai [4]):

- (m) $e_0 |D = \log |z - \zeta| + w(z)$ with $w \in H(\bar{D})$ and $w(\zeta) = 0$,
- (n) e_0 is of L_1 -type at δ ,
- (o) $\lim_{z \rightarrow \gamma} e_0(z) = +\infty$.

Needless to say $e_0 \in F$.

7. Proof of theorem. Consider $p_\lambda = \lambda e_0 + (1 - \lambda)q$ with a fixed $q \in F$ and $0 < \lambda < 1$. It is clear that $\lim_{\lambda \rightarrow \gamma} p_\lambda(z) = +\infty$ and $p_\lambda \in F$. Therefore by Lemma 6 we obtain

$$(5) \quad \{p_\lambda\}_{0 < \lambda < 1} \subset C_\gamma.$$

On the other hand, obviously

$$(6) \quad C_\gamma \subset F.$$

Moreover observe that $\lambda \rightarrow p_\lambda$ is injective if $e_0 \neq q$.

By the approximation theorem of Heins [2], we can see at once that if $d_\gamma = 1$, so is the cardinal number of F . It is trivial that the converse is valid. Hence $c_\gamma = 1$ if and only if $d_\gamma = 1$.

Suppose that $d_\gamma \geq 2$. Then there exists a $q \in F$ with $q \neq e_0$. By the injectivity of $\lambda \rightarrow p_\lambda$, $\psi \leq c_\gamma$. Conversely it follows from (6) that $c_\gamma \leq$ the cardinal number of F which is not greater than ψ . Thus $c_\gamma = \psi$. In either case, since $d_\gamma \leq \psi$, we have $c_\gamma = 1 + (d_\gamma - 1)\psi$.

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