ON INTERPOLATION OF $q$-VARIATE STATIONARY STOCHASTIC PROCESSES

HABIB SALEHI
ON INTERPOLATION OF q-VARIATE STATIONARY STOCHASTIC PROCESSES

HABIB SALEHI

Let $X_t$ be a q-variate stationary stochastic process. Let $K$ be any set of $t$-values and let $K'$ be the complement of $K$. If $s \in K'$ the problem of approximating $X_t$ by linear combinations of the $X_t$'s with $t \in K$ and limit of such linear combinations is considered. The best linear predictor and the mean square error matrix are evaluated in the following cases: (1) $t$ takes on all real values, $K$ consists of the integers (2) $t$ is integer-valued, $K$ consists of the odd integers.

Let $(X_k)_{-\infty}^{+\infty}, k$ an integer, be a q-variate weakly stationary stochastic process (SP). Let $K$ be any subset of the set of integers and $K'$ denote its complement in the set of all integers. Let $\mathcal{M}_K$ denote the (closed) subspace spanned by $X_k, k \in K$.

Prediction Problem. Let $X_s, s \in K'$. Find $\hat{X}_s$ the projection of $X_s$ onto $\mathcal{M}_K$ and the error matrix $(X_s - \hat{X}_s, X_s - \hat{X}_s)^t$.

In this paper we propose to solve the prediction problem for two cases:

1) $X_t, t$ real, is a q-variate stationary SP and $K$ consists of the set of all integers.

2) $X_k, k$ an integer, is a q-variate stationary SP and $K$ consists of the set of all odd integers.

For $q = 1$ these results have been previously obtained by A. M. Yaglom [cf. [12, p. 176]].

In § 2 we will review the notion of absolute continuity of a matrix-valued signed measure with respect to another such measure [cf. [6]] and state a few results concerning the Hellinger-square integrability of matrix-valued measures. Our main result will be given in § 3.

2. Matrix-valued measures. The problem of absolute continuity of a matrix-valued measure with respect to another matrix-valued measure was first posed by P. Masani in [4, p. 366]. Later J. B. Robertson and M. Rosenberg [cf. [6]] dealt with this question and were able to obtain a satisfactory solution to it. We will briefly review some of these results. Let $\Omega$ be any set and $\mathcal{B}$ be a $\sigma$-algebra of its subsets. $M$ is said to be a $q \times r$ matrix-valued signed measure on $(\Omega, \mathcal{B})$ if for each $B \in \mathcal{B}$, $M(B)$ is a $q \times r$ matrix, with finite complex

\[^1\) (...) denotes the inner product in the Hilbert space $\mathcal{H}^q$ containing the q-variate stochastic process $X_k, k$ an integer.
entries, and \( M(B) = \sum_{i=1}^{\infty} M(B_i) \), whenever \( B_1, B_2, \ldots \) is a sequence of disjoint sets in \( \mathcal{B} \) whose union is \( B \). A \( p \times q \) matrix-valued signed measure \( M \) is called a \( p \times q \) matrix-valued measure if \( M(B) \) is a nonnegative hermitian matrix for each \( B \in \mathcal{B} \). \( \Psi \) is called a measurable \( p \times q \) matrix-valued function on \((\Omega, \mathcal{B})\) if for each \( \omega \in \Omega \), \( \Psi(\omega) \) is a \( p \times q \) matrix and if the entries of \( \Psi \) are measurable functions on \((\Omega, \mathcal{B})\). We say that a \( q \times r \) matrix-valued signed measure is absolutely continuous (a.c.) with respect to (w.r.t.) a \( \sigma \)-finite nonnegative real-valued measure \( \mu \). \( \Psi \) is called a measurable \( p \times q \) matrix-valued function on \((\Omega, \mathcal{B})\) if for each \( \omega \in \Omega \), \( \Psi(\omega) \) is a \( p \times q \) matrix and if the entries of \( \Psi \) are measurable functions on \((\Omega, \mathcal{B})\). The integral \( N(B) = \int_B \Psi \, dM \) is defined by 

\[
N(B) = \int_B \Psi \, dM = \int_B \Psi \, \frac{dM}{d\mu} \, d\mu,
\]

where \( \frac{dM}{d\mu} \) is the Radon-Nikodym derivative of \( M \) w.r.t. \( \mu \). It is easy to show that the definition of \( N(B) \) is independent of the choice of \( \mu \).

**Definition 2.1.** Let \( M \) and \( N \) be \( p \times q \) and \( r \times q \) matrix-valued signed measures on \((\Omega, \mathcal{B})\) respectively, \( \mu \) be any \( \sigma \)-finite nonnegative real-valued measure on \((\Omega, \mathcal{B})\) such that \( M \) and \( N \) are a.c. w.r.t. \( \mu \). We say that \( N \) is a.c. w.r.t. \( M \) if

\[
\kappa\left( \frac{dM}{d\mu} (\omega) \right) \subset \kappa\left( \frac{dN}{d\mu} (\omega) \right) \quad \text{a.e.} \quad \mu,
\]

where for each matrix \( A \), \( \kappa(A) = \{ \alpha \in \mathbb{C} : A\alpha = 0 \} \). It can be easily verified that this definition is independent of \( \mu \).

The following theorem is proved in [6].

**Theorem 2.2.** Let \( M \) and \( N \) be \( p \times q \) and \( r \times q \) matrix-valued signed measures on \((\Omega, \mathcal{B})\). Then

(a) \( N \) is a.c. w.r.t. \( M \) if and only if there exists a measurable \( r \times p \) matrix-valued function \( \Psi \) on \( \Omega \) such that for each \( B \in \mathcal{B} \)

\[
N(B) = \int_B \Psi \, dM.
\]

(b) Let \( \Phi \) and \( \Psi \) be measurable \( r \times p \) matrix-valued functions on \( \Omega \). Then for each \( B \in \mathcal{B} \),

\[
\int_B \Phi \, dM = \int_B \Psi \, dM
\]

a.e. \( \mu \), where \( J \) is the orthogonal projection matrix onto the range of \( \frac{dM}{d\mu} \) and \( \mu \) is any \( \sigma \)-finite nonnegative real-valued measure on \((\Omega, \mathcal{B})\) w.r.t. which \( M \) is a.c.

If \( N \) is a.c. w.r.t. \( M \), then by Theorem 2.2 (a) there exists a measurable matrix-valued function \( \Psi \) such that for each \( B \in \mathcal{B} \)

\[
N(B) = \int_B \Psi \, dM.
\]
Ψ is called the Radon-Nikodym derivative of N w.r.t. M and we will denote it by \( (dN/dM) \). We now review properties of Hellinger integrability of matrix-valued measures [cf. [9]].

**Definition 2.3.** Let \( M \) and \( N \) be \( p \times q \) and \( r \times q \) be matrix-valued measures on \( (\Omega, \mathcal{B}) \), \( F \) be a \( q \times q \) matrix-valued measure on \( (\Omega, \mathcal{B}) \). We say that \( (M, N) \) is Hellinger-integrable w.r.t. \( F \) if \[
\int_{\Omega} \left( \frac{dM}{d\mu} \frac{dF}{d\mu} \right)^{-1} \left( dN/d\mu \right)^* d\mu^2
\]
exists for some \( \sigma \)-finite nonnegative real-valued measure on \( (\Omega, \mathcal{B}) \), where \( (dF/d\mu)^{-1} \) denotes the generalized inverse of \( (dF/d\mu) \) [cf. [5, p. 407]]. It is not hard to show that the existence and the value of this integral when it exists is independent of \( \mu \). We write

\[
\int_{\Omega} \frac{dM}{dF} N^* = \int_{\Omega} \left( \frac{dM}{d\mu} \frac{dF}{d\mu} \right)^{-1} \left( dN/d\mu \right)^* d\mu.
\]

The following theorem is needed later.

**Theorem 2.4.** Let (i) \( M \) and \( N \) be \( p \times q \) and \( r \times q \) matrix-valued signed measures on \( (\Omega, \mathcal{B}) \), \( F \) be a \( q \times q \) matrix-valued measure on \( (\Omega, \mathcal{B}) \).

(ii) \( M \) or \( N \), say \( M \), be a.c. w.r.t. \( F \). Then \( (M, N) \) is Hellinger integrable w.r.t. \( F \) if and only if the Lebesgue integral \( \int_{\Omega} (dM/dF) dN^* \) exists. In case these integrals exist, their values are equal.

**Proof.** Let \( \mu \) be any \( \sigma \)-finite nonnegative real-valued measure on \( (\Omega, \mathcal{B}) \) w.r.t. which \( M, N \) and \( F \) are a.e. Since \( M \) is a.c. w.r.t. \( F \) then by Theorem 2.2 there exists a measurable \( p \times q \) matrix-valued function \( \Psi \) on \( \Omega \) such that for each \( B \in \mathcal{B} \)

\[
(1) \quad M(B) = \int_{B} \Psi dF, \Psi J = \Psi \text{ a.e. } \mu,
\]

where \( J \) is the orthogonal projection matrix onto the range of \( dF/d\mu \). If \( \int_{\Omega} dM dN^*/dF \) exists, then from the following chain of equality it follows that \( \int_{\Omega} (dM/dF) dN^* \) exists and the two integrals are equal

\[
\int_{\Omega} \frac{dM}{dF} N^* = \int_{\Omega} \left( \frac{dM}{d\mu} \frac{dF}{d\mu} \right)^{-1} \left( dN/d\mu \right)^* d\mu
\]

\[
= \int_{\Omega} \Psi \left( dF/d\mu \right)^{-1} \left( dN/d\mu \right)^* d\mu \]

\[
= \int_{\Omega} \Psi (dN/d\mu)^* d\mu = \int_{\Omega} \left( dM/dF \right) dN^*,
\]

\( ^2 \) denotes the adjoint operation.
where the first equality is a consequence of Definition 2.3, the second is a consequence of (1), the third one is a consequence of \((dF/d\mu)(dF/d\mu)^* = J\) and (1) and the last two are consequences of (1). Similarly if \(\int_0^1 (dM/dF)dN^*\) exists from (2) it follows that \(\int_0^1 dMdN^*/dF\) exists and these integrals are equal.

3. Interpolation of a stationary SP with continuous time parameter. Let \(X_t, t \text{ real},\) be a \(q\)-variate weakly stationary SP with the spectral distribution \(q \times q\) matrix-valued function \(F\) defined on \((-\infty, \infty)\). Suppose that the process has been observed at the time points \(k = \cdots, -1, 0, 1, \cdots\) and we wish to estimate \(X_t\) where \(t\) is not an integer. First we state a lemma whose proof is immediate.

**Lemma 3.1.** Let \(K\) be the set of all integers. Then
(a) for each \(\lambda \in (0, 2\pi]\) the series
\[
\sum_{k \in K} [F(\lambda + 2k\pi) - F(2k\pi)]
\]
converges and defines a \(q \times q\) nonnegative hermitian matrix-valued function \(G(\cdot)\) on \((0, 2\pi]\).
(b) \(G(\cdot)\) is monotone nondecreasing on \((0, 2\pi]\) and
\[
G(2\pi) = \lim_{\lambda \to 2\pi} F(\lambda)
\]
(c) For each \(\lambda \in (0, \pi]\) and each fixed real \(t\) the series
\[
\sum_{k \in K} e^{-2\pi k\pi t}[F(\lambda + 2k\pi) - F(2k\pi)]
\]
converges and defines a \(q \times q\) matrix-valued function \(G_t(\cdot)\) on \((0, 2\pi]\).
(d) \(G_t\) is of bounded variation on \((0, 2\pi]\) and the variation of \(G_t \leq G(2\pi)\).
(e) \(G\) and \(G_t\) define \(q \times q\) matrix-valued measure and signed measure on the Borel family of subsets of \((0, \pi]\) respectively.
(f) \(G_t\) is a.c. w.r.t. \(G^3\)

We are now ready to state the main result of this notion. For standard terminology and notation of \(q\)-variate stationary processes used in Theorem 3.2 we refer to [4] and [8].

**Theorem 3.2.** (i) Let \(X_t, t \text{ real},\) be a \(q\)-variate weakly stationary SP with the spectral representation \(X_t = \int_{-\infty}^\infty e^{-it\lambda} E(d\lambda)X_0,\) the spectral

\(3\) By "\(G_t\) is a.c. w.r.t. \(G\)" we mean that the \(q \times q\) matrix-valued signed measure \(M_t\) generated by \(G_t\) is absolutely continuous w.r.t. the \(q \times q\) matrix-valued measure \(M\) generated by \(G\).
distribution function $F$ defined on $(-\infty, \infty)$.

(ii) Let $K$ denote the set of all integers, $\mathcal{M}_K$ the (closed) subspace spanned by $X_t$, $t \in K$ and for each $t \in K$ let $\hat{X}_t$ be the projection of $X_t$ onto $\mathcal{M}_K$. Then

(a) There exists a $q \times q$ matrix-valued function $\Psi_t \in L_{2, F}$ such that $\hat{X}_t = \int_{-\infty}^{\infty} \Psi_t(\lambda) E(d\lambda) X_0$, the function $\Psi_t$ is periodic of period $2\pi$.

(b) If $G(\cdot)$ and $G_t(\cdot)$ are the matrix-valued functions defined in Lemma 3.1, then

\[ \Psi_t(\lambda) = e^{-it\lambda} (dG_t/dG)(\lambda) \quad \text{a.e. } F. \]

(c) The interpolation error matrix $\Sigma_t = (X_t - \hat{X}_t, X_t - \hat{X}_t)$ is given by

\[ \Sigma_t = \frac{1}{2\pi} \int_0^{2\pi} (I - dG_t/dG) dF(I - dG_t/dG)^*, \]

where $I$ is the identity matrix of order $q \times q$.

**Proof.** (a) Let $V$ denote the isomorphism mapping from $L_{2, F}$ onto $\mathcal{M}$ the (closed) subspace spanned by the SP $X_t$ [cf. [7, p. 297]]. Since $\mathcal{M}_K \subseteq \mathcal{M}$, there exists a $\Psi_t \in L_{2, F}$ such that

\[ \hat{X}_t = \int_{-\infty}^{\infty} \Psi_t(\lambda) E(d\lambda) X_0. \]

From the definition of $V$ it follows that for each $k \in K$

\[ Ve^{-ik\lambda} I = X_k. \]

Since for each $k \in K$, $e^{-ik\lambda}$ has period $2\pi$ and since $\hat{X}_t \in \mathcal{M}_K$, from (1) and (2) it follows that $\Psi_t(\lambda)$ is periodic and has period $2\pi$.

(b) By (a) we have

\[ \hat{X}_t = \int_{-\infty}^{\infty} \Psi_t(\lambda) E(d\lambda) X_0. \]

It then immediately follows that

\[ \int_{-\infty}^{\infty} \left[ e^{-it\lambda} I - \Psi_t(\lambda) dF(\lambda) e^{-ik\lambda} \right] (X_t - \hat{X}_t, X_k) = 0 \]

for each $k \in K$.

Since $\Psi_t \in L_{2, F}$, $\Psi_t \in L_{2, G} \cap L_{2, G_t}$. Hence

\[ \int_0^{2\pi} e^{-ik\lambda} [e^{-it\lambda} dG_t(\lambda) - \Psi_t(\lambda) dG(\lambda)] = \int_0^{2\pi} e^{-ik\lambda} e^{-it\lambda} dG_t(\lambda) - \int_0^{2\pi} e^{-ik\lambda} \Psi_t(\lambda) dG(\lambda). \]

---

4 $L_{2, F}$ is an abbreviation for $L_2((-\infty, \infty), \mathcal{B}, F)$, [cf. [7, p. 295]].
The first term = \[\int_0^{2\pi} e^{-ik_1}d\left(\sum_{n \in K} e^{-2in\pi t}\left[F(\lambda + 2n\pi) - F(2n\pi)\right]\right)\]

= \[\sum_{n \in K} \int_0^{2\pi} e^{-ik_1}e^{-i\lambda t}d(e^{-2in\pi t}[F(\lambda + 2n\pi) - F(2n\pi)])\]

= \[\sum_{n \in K} \int_{2n\pi}^{2(n+1)\pi} e^{-ik(x-2n\pi)}e^{-i\lambda t}d[F(\mu) - F(2n\pi)]\]

= \[\sum_{n \in K} \int_{2n\pi}^{2(n+1)\pi} e^{-ik_1e^{it\lambda}}d[F(\mu) - F(2n\pi)]\]

= \[\int_{-\infty}^{\infty} e^{-it\lambda}e^{-ik_1}dF(\lambda)\] .

Also since \(T_\lambda(\lambda)\) is periodic of period \(2\pi\),

\[\int_0^{2\pi} e^{-ik_1\Psi_\lambda(\lambda)}dG(\lambda) = \int_0^{2\pi} e^{-ik_1\Psi_\lambda(\lambda)}d\left[\sum_{n \in K} F(\lambda + 2n\pi) - F(2n\pi)\right]\]

= \[\sum_{n \in K} \int_0^{2\pi} e^{-ik_1\Psi_\lambda(\lambda)}d[F(\lambda + 2n\pi) - F(2n\pi)]\]

= \[\sum_{n \in K} \int_{2n\pi}^{2(n+1)\pi} e^{-ik_1\Psi_\lambda(\lambda)}d[F(\lambda) - F(2n\pi)]\]

= \[\int_{-\infty}^{\infty} e^{-ik_1\Psi_\lambda(\lambda)}dF(\lambda)\] .

Hence

\[\int_0^{2\pi} e^{-ik_1}[e^{-it\lambda}dG_\lambda(\lambda) - \Psi_\lambda(\lambda)]dG(\lambda)\]

= \[\int_{-\infty}^{\infty} [e^{-it\lambda}I - \Psi_\lambda(\lambda)]e^{-ik_1}dF(\lambda)\] .

By (3) and (4) we get that

\[\int_0^{2\pi} e^{-ik_1}\Psi_\lambda(\lambda)dG(\lambda) = \int_0^{2\pi} e^{-ik_1\Psi_\lambda(\lambda)}dG(\lambda)\] .

Since by (5) the Fourier coefficients of the matrix-valued signed measures 
\(M(B) = \int_B e^{-it\lambda}dG_\lambda(\lambda)\) and 
\(N(B) = \int_B \Psi_\lambda(\lambda)dG(\lambda)\), 
\(B\) is a Borel subset of \((0, 2\pi]\), are the same, it follows that for each Borel subset \(B\) of \((0, 2\pi]\)

\[M(B) = \int_B e^{-it\lambda}dG_\lambda(\lambda) = \int_B \Psi_\lambda(\lambda)dG(\lambda)\] .

Now let \(\mu\) be any \(\sigma\)-finite nonnegative real-valued measure on \((\Omega, \mathcal{B})\) 
w.r.t. \(G\) is a.c. Then automatically \(G_i\) is a.c. w.r.t. \(\mu\), because \(G_i\) is a.c. w.r.t. \(G\). Therefore we have

\[M(B) = \int_B e^{-it\lambda}(dG_i/dG)(\lambda)dG(\lambda) = \int_B \Psi_\lambda(\lambda)dG(\lambda)\] .

From (6) and Theorem 2.2 (b) it follows that
where \( J \) is the orthogonal projection matrix onto the range of \( dG/d\mu \).

Since \( G \) is a.c. w.r.t. \( \mu \), \( F' \) is also a.c. w.r.t. \( \mu \). Because \( \Psi_t \in L_{a,F} \) a simple calculation shows that \( \Psi_t J = L_{a,F} \) and that

\[
\Psi_t J = \Psi_t \quad \text{a.e.} \quad \mu.
\]

But \( (dG_t/dG)J = \Psi_t J \), therefore \( (dG_t/dG)J \in L_{a,F} \). This easily implies that \( (dG_t/dG) \in L_{a,F} \) and

\[
(dG_t/dG)J = (dG_t/dG) \quad \text{a.e.} \quad \mu.
\]

From (7), (8) and (9) we have

\[
e^{-it\lambda}(dG_t/dG) = \Psi_t \quad \text{a.e.} \quad \mu
\]

i.e.

\[
e^{-it\lambda}(dG_t/dG) = \Psi_t, \quad \text{a.e.} \quad F'.
\]

(c) We have \( X_t = \int_{-\infty}^{\infty} e^{-it\lambda}E(d\lambda)X_0 \) and

\[
\hat{X}_t = \int_{-\infty}^{\infty} e^{-it\lambda}(dG_t/dG)(\lambda)E(d\lambda)X_0.
\]

Hence from the isometry theorem [cf. [7, p. 297]] we obtain

\[
\Sigma_t = (X_t - \hat{X}_t, X - \hat{X}_t) = \frac{1}{2\pi} \int_0^{2\pi} (I - dG_t/dG)dF(I - dG_t/dG)^*.
\]

As a special case of Theorem 3.2 we have the following result concerning a \( q \)-variate stationary stochastic process with discrete time parameter.

**Theorem 3.3.** Let

(i) \( X_k, k \) an integer, be a \( q \)-variate weakly stationary SP with the spectral representation \( X_k = \int_0^{2\pi} e^{-ik\lambda}E(d\lambda)X_0 \) with spectral distribution \( F \) defined on \( (0, 2\pi] \).

(ii) Let \( K \) be the set of all odd integers, \( \mathcal{A}_K \) the (closed) subspace spanned by \( X_k, k \in K \) and let for each \( k \in K \), \( \hat{X}_k \) denote the projection of \( X_k \) onto \( \mathcal{A}_K \). Then

(a) there exists a \( q \times q \) matrix-valued function \( \Psi_k \in L_{a,F} \) such that \( \hat{X}_k = \int_0^{2\pi} \Psi_k(\lambda)E(d\lambda)X_0 \). \( e^{it\lambda} \) is periodic of period \( \pi \).

(b) \( \Psi_k \) is given by

\[
\Psi_k(\lambda) = e^{-ik\lambda} \frac{d[F(\cdot) + e^{-it\lambda}F(\cdot + \pi)]}{d[F(\cdot) + F(\cdot + \pi)]}(\lambda) \quad \text{a.e.} \quad F \text{ if } \lambda \in (0, \pi]
\]
\[ \psi_k(\chi) = e^{-i(k+1)\pi} \psi_k(X - \pi) \text{ a.e. } F \text{ if } \lambda \in (\pi, 2\pi]. \]

(c) The interpolation error matrix \( \Sigma_k = (X - \hat{X}_k, X - \hat{X}_k) \) is given by

\[
\Sigma_k = \frac{2}{\pi} \int_0^{\pi} \frac{dF(\lambda + \pi)}{d[F(\lambda) + F(\lambda + \pi)]} dF(\lambda),
\]

where the first is a Lebesgue integral and the last one is a Hellinger integral.

**Proof.** Since the proof of (a) is similar to that of Theorem 3.2 (a), we proceed to sketch the proof of parts (b) and (c). Let for each real \( t \)

\[
S(t) = \int_t^{t+\pi} \exp \left\{ -i \left( t - \frac{1}{2} \right) \lambda \right\} dF(\frac{\lambda}{2})
\]

and \( Y(t) \) be a \( q \)-variate stationary stochastic process with correlation function \( S(t) \). Note that for each integer \( n \)

\[ S(n/2) = R(n - 1). \]

Using results (b) and (c) of Theorem 3.2 for the processes \( Y(t) \), from (1), part (b) and the first equation for \( \Sigma_k \) easily follow. The second equation for \( \Sigma_k \) is obtained from Theorem 2.4, since \( dF(\lambda + \pi) \) is a.e. w.r.t. \( d[F(\lambda) + F(\lambda + \pi)] \).

**References**


Received October 23, 1967. This research was partially supported by NSF GP-7535 and GP-8614.

MICHIGAN STATE UNIVERSITY
<table>
<thead>
<tr>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Patrick Robert Ahern, <em>On the geometry of the unit ball in the space of real annihilating measures</em></td>
<td>1</td>
</tr>
<tr>
<td>Kirby Alan Baker, <em>Equational classes of modular lattices</em></td>
<td>9</td>
</tr>
<tr>
<td>E. F. Beckenbach and Gerald Andrew Hutchison, <em>Meromorphic minimal surfaces</em></td>
<td>17</td>
</tr>
<tr>
<td>Tae Ho Choe, <em>Intrinsic topologies in a topological lattice</em></td>
<td>49</td>
</tr>
<tr>
<td>John Bligh Conway, <em>A theorem on sequential convergence of measures and some applications</em></td>
<td>53</td>
</tr>
<tr>
<td>Roger Cuppens, <em>On the decomposition of infinitely divisible probability laws without normal factor</em></td>
<td>61</td>
</tr>
<tr>
<td>Lynn Harry Erbe, <em>Nonoscillatory solutions of second order nonlinear differential equations</em></td>
<td>77</td>
</tr>
<tr>
<td>Burton I. Fein, <em>The Schur index for projective representations of finite groups</em></td>
<td>87</td>
</tr>
<tr>
<td>Stanley P. Gudder, <em>A note on proposition observables</em></td>
<td>101</td>
</tr>
<tr>
<td>Kenneth Kapp, <em>On Croisot’s theory of decompositions</em></td>
<td>105</td>
</tr>
<tr>
<td>Robert P. Kaufman, <em>Gap series and an example to Malliavin’s theorem</em></td>
<td>117</td>
</tr>
<tr>
<td>E. J. McShane, Robert Breckenridge Warfield, Jr. and V. M. Warfield, <em>Invariant extensions of linear functionals, with applications to measures and stochastic processes</em></td>
<td>121</td>
</tr>
<tr>
<td>Marvin Victor Mielke, <em>Rearrangement of spherical modifications</em></td>
<td>143</td>
</tr>
<tr>
<td>Akio Osada, <em>On unicity of capacity functions</em></td>
<td>151</td>
</tr>
<tr>
<td>Donald Steven Passman, <em>Some 5/2 transitive permutation groups</em></td>
<td>157</td>
</tr>
<tr>
<td>Harold L. Peterson, Jr., <em>Regular and irregular measures on groups and dyadic spaces</em></td>
<td>173</td>
</tr>
<tr>
<td>Habib Salehi, <em>On interpolation of q-variate stationary stochastic processes</em></td>
<td>183</td>
</tr>
<tr>
<td>Michael Samuel Skaff, <em>Vector valued Orlicz spaces generalized N-functions. I</em></td>
<td>193</td>
</tr>
<tr>
<td>Thomas Paul Whaley, <em>Algebras satisfying the descending chain condition for subalgebras</em></td>
<td>217</td>
</tr>
<tr>
<td>G. K. White, <em>On subgroups of fixed index</em></td>
<td>225</td>
</tr>
<tr>
<td>Martin Michael Zuckerman, <em>A unifying condition for implications among the axioms of choice for finite sets</em></td>
<td>233</td>
</tr>
</tbody>
</table>