VECTOR VALUED ORLICZ SPACES GENERALIZED N-FUNCTIONS. I

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The theory of Orlicz spaces generated by $N$-functions of a real variable is well known. On the other hand, as was pointed out by Wang, this same theory generated by $N$-functions of more than one real variable has not been discussed in the literature. The purpose of this paper is to develop and study such a class of generalized $N$-functions (called $GN$-functions) which are a natural generalization of the functions studied by Wang and the variable $N$-functions by Portnov. In second part of this study we will utilize $GN$-functions to define vector-valued Orlicz spaces and examine the resulting theory.

This paper is divided into five sections. In §2, we define and examine some basic properties of $GN$-functions. A generalized delta condition is introduced and characterized in §3. In §4 and §5 we present, respectively, the theory of an integral mean for $GN$-functions and the concept of a conjugate $GN$-function. A complete bibliography on Orlicz spaces, $N$-functions, and related material can be found in [4, 8]. The study of variable $N$-functions by Portnov can be found in [6, 7] and the study of nondecreasing $N$-functions by Wang in [9].

2. $GN$-functions. In what follows $T$ will denote a space of points with $\sigma$-finite measure and $E^n$ $n$ dimensional Euclidean space.

**Definition 2.1.** Let $M(t, x)$ be a real valued nonnegative function defined on $T \times E^n$ such that

(i) $M(t, x) = 0$ if and only if $x = 0$ for all $t \in T, x \in E^n$,

(ii) $M(t, x)$ is a continuous convex function of $x$ for each $t$ and a measurable function of $t$ for each $x$,

(iii) For each $t \in T$, $\lim_{|x| \to \infty} \frac{M(t, x)}{|x|} = \infty$, and

(iv) There is a constant $d \geq 0$ such that

\[ \inf_t \inf_{c \in d} k(t, c) > 0 \]

where

\[ k(t, c) = \frac{M(t, c)}{\bar{M}(t, c)}, \quad \bar{M}(t, c) = \sup_{|x| = c} M(t, x), \]

\[ \bar{M}(t, c) = \inf_{|x| = c} M(t, x) \]
and if \( d > 0 \), then \( \tilde{M}(t, d) \) is an integrable function of \( t \). We call a function satisfying properties (i)—(iv) a generalized \( N \)-function or a GN-function.

GN-functions are coordinate independent and are not necessarily symmetric. Therefore, such functions as \( M(t, x) = x_1^2 + x_2^2 + (x_1 - x_2)^3 \) which are not nondecreasing (as defined in [9]) are allowed in the class of GN-functions. The next theorem illustrates this point.

**Theorem 2.1.** If \( M(t, x) \) is a GN-function and \( A \) is an orthogonal linear transformation defined on \( E^n \) with range in \( E^n \), then \( \tilde{M}(t, x) = M(t, Ax) \) is a GN-function.

Properties (i)—(iv) when applied to \( \tilde{M}(t, x) \) follow immediately from the same properties for \( \Lambda f(\tau, x) \) (see [8, Th. 8.1]).

The next theorem characterizes a part of property (iv) in Definition 2.1 and provides a means of comparing function values at different points for GN-functions when \( |x| \) is large.

**Theorem 2.2.** A necessary and sufficient condition that (*) hold is that if \( |x| \leq |y| \), then there exist constants \( K \geq 1 \) and \( d \geq 0 \) such that \( M(t, x) \leq KM(t, y) \) for each \( t \in T \) and \( |x| \geq d \).

If (*) is true, then there exists a constant \( d \geq 0 \) such that \( l(t) = \inf_{c \leq d} k(t, c) > 0 \) for each \( t \) in \( T \). By definition of \( k(t, c) \) this means

\[
(2.2.1) \quad M(t, y) \geq M(t, |y|) \geq l(t)\tilde{M}(t, |y|)
\]

for any \( y \) such that \( |y| = c \geq d \). On the other hand, if \( d \leq |x| \leq |y| \), then the convexity of \( M(t, x) \) and \( M(t, 0) = 0 \) yields

\[
(2.2.2) \quad \tilde{M}(t, |y|) \geq \sup_{|z|=|y|} M(t, z).
\]

Combining (2.2.1) and (2.2.2) we arrive at

\[
M(t, y) \geq l(t) \sup_{|z|=|y|} M(t, z) \geq K^{-1}M(t, x)
\]

whenever \( d \leq |x| \leq |y| \) where \( K^{-1} = \inf_l l(t) > 0 \).

The converse follows easily from the condition in the theorem.

It is interesting to note that if \( M(t, x) \) is a GN-function, then \( 2\tilde{M}(t, x) = M(t, x) + \tilde{M}(t, x) \) is also a GN-function where \( \tilde{M}(t, x) \) is defined as in Theorem 2.1. This means we can construct a symmetric (in \( x \)) GN-function from one which does not possess this property. For, if \( \tilde{M}(t, x) = M(t, -x) \), then \( \tilde{M}(t, x) \) is clearly symmetric in \( x \).

Property (iv) of Definition 2.1 provides the condition which allows
a natural generalization from $N$-functions of a real variable to those of several real variables. Let us observe that the function $\widetilde{M}(t, c)$ is also a $GN$-function of a real nonnegative variable $c$. On the other hand, $M(t, c)$ need not even be convex in $c$.

Since $\widetilde{M}(t, c) \leq M(t, x) \leq \widetilde{M}(t, c)$ for each $x$ such that $|x| = c$, we would like to find a $GN$-function which bounds $M(t, c)$ from below for all $c$. If $d = 0$ in Theorem 2.2, then $K^{-1}M(t, c)$ would do.

One might accomplish the construction of such a function by taking the supremum of a class of convex functions bounding $M(t, c)$ from below. This function would be convex. However, this class may be empty. The next theorem shows that this is not the case whenever $M(t, x)$ is a $GN$-function. The construction employed can be applied to more general settings than exist here.

**Theorem 2.3.** If $M(t, x)$ is a $GN$-function and $M(t, c)$ is defined as above, then there exists a $GN$-function $R(t, c)$ such that $R(t, c) \leq M(t, c)$ for all $c \geq 0$.

Since $M(t, c)$ satisfies property (iii) of Definition 2.1, given any $d > 0$ there is a $c_0 > 0$ such that $M(t, c) \geq dc$ whenever $c \geq c_0$. Let us define the function

$$P(t, c) = \begin{cases} \sup_{0 < w \leq 1} \frac{M(t, cw)}{w} & \text{if } c \geq c_0 \\ \frac{M(t, c)}{c} & \text{if } 0 \leq c < c_0 \end{cases}$$

Then it is easy to show that (i) $P(t, ac) \leq aP(t, c)$ for $0 \leq a \leq 1$, (ii) \{P(t, c)/c\} is a nondecreasing function of $c$, and (iii) $P(t, c)$ is finite for each $c$. We now obtain the desired function $R(t, c)$ by defining

$$R(t, c) = \int_0^c Q(t, s)ds$$

where

$$Q(t, c) = \begin{cases} \frac{P(t, c)}{c} & \text{if } c \geq c_0 \\ \frac{cP(t, c_0)}{c_0^2} & \text{if } 0 \leq c < c_0 \end{cases}$$

We have immediately that

$$R(t, c) \leq cQ(t, c) = P(t, c) \leq M(t, c) \, .$$

If is not difficult to show that $R(t, c)$ is also a $GN$-function.
3. Delta condition. In this section a generalized growth condition is defined for \( GN \)-functions. This growth or delta condition generalizes that definition usually given for a real variable \( N \)-function (e.g., see [4, 6, 7]).

**Definition 3.1.** We say a \( GN \)-function \( M(t, x) \) satisfies a \( \Delta \)-condition if there exist a constant \( K \geq 2 \) and a non-negative measurable function \( \delta(t) \) such that the function \( \overline{M}(t, 2\delta(t)) \) is integrable over the domain \( T \) and such that for almost all \( t \) in \( T \) we have

\[
(*) \quad M(t, 2x) \leq KM(t, x)
\]

for all \( x \) satisfying \( |x| \geq \delta(t) \).

We say a \( GN \)-function satisfies a \( \Delta_0 \)-condition if it satisfies a \( \Delta \)-condition with \( \delta(t) = 0 \) for almost all \( t \) in \( T \).

In Definition 3.1 we could have used any constant \( l > 1 \) in place of the scalar 2 in (**)'. This is the basis of the next theorem which gives an equivalent definition to that employed in 3.1.

**Theorem 3.1.** A \( GN \)-function \( M(t, x) \) satisfies a \( \Delta \)-condition if and only if given any \( l > 1 \) there exists a constant \( K \geq 2 \) and a nonnegative measurable function \( \delta(t) \) such that \( \overline{M}(t, 2\delta(t)) \) is integrable over \( T \) and such that for almost all \( t \) in \( T \) we have

\[
(3.1.1) \quad M(t, lx) \leq K_lM(t, x)
\]

whenever \( |x| \geq \delta(t) \).

Suppose \( M(t, x) \) satisfies a \( \Delta \)-condition and \( l > 1 \). We choose \( m \) so large that \( 2^m \geq l \). Then by convexity and our assumption of a \( \Delta \)-condition there is a \( K \geq 2 \) and measurable \( \delta(t) \geq 0 \) such that for almost all \( t \) in \( T \)

\[
M(t, lx) \leq M(t, 2^m x) \leq K^m M(t, x)
\]

whenever \( |x| \geq \delta(t) \). Therefore (3.1.1) holds with \( K_l = K^m \). The converse follows as easily.

Whenever we deal with convex functions of several variables the concept of a one sided directional derivative plays an important role. The next result utilizes such a function, so we define it now.

**Definition 3.2.** For each \( t \) in \( T \) the directional derivative of a \( GN \)-function \( M(t, x) \) in a direction \( y \) is defined by

\[
M'(t, x; y) = \lim_{h \to 0^+} \frac{M(t, x + hy) - M(t, x)}{h}
\]
The important properties of directional derivatives of convex functions of several variables which will be needed can be found in [3, 8]. Using the directional derivative defined above, the next result characterizes the delta condition and generalizes similar results given in [4, 6, 7].

**Theorem 3.2.** A GN-function $M(t, x)$ satisfies a $\Delta$-condition if and only if there exists a nonnegative measurable function $\delta(t)$ such that $M(t, 2\delta(t))$ is integrable over $T$ and a constant $c > 1$ such that for almost all $t$ in $T$

$$\frac{M'(t, x; x)}{M(t, x)} < c$$

whenever $|x| \geq \delta(t)$. Moreover, if (3.2.1) holds, then for almost all $t$ in $T$ and for each $x$ such that $|x| \geq \delta(t)$ we have

$$M(t, px) < M(t, x)p$$

for all $p > 1$.

Suppose $M(t, x)$ satisfies a $\Delta$-condition. Then, by convexity (see, [8, Th. 5.3]), we must have for some $K \geq 2$ and $\delta(t) \geq 0$

$$KM(t, x) \geq M(t, 2x) \geq M(t, x) + M'(t, x; x)$$

whenever $|x| \geq \delta(t)$. This means (3.2.1) holds with $c = K$.

Conversely, suppose (3.2.1) holds. We choose $s$ such that $s \geq 1$. Then, by assumption, there is a constant $c > 1$ and $\delta(t) > 0$ such that for almost all $t$ in $T$

$$\frac{M'(t, sx; sx)}{M(t, sx)} > c$$

whenever $|x| \geq \delta(t)$. On the other hand, we have

$$\frac{d}{ds} M(t, sx) = \lim_{h \to 0^+} \frac{M(t, sx + hx) - M(t, sx)}{h} = M'(t, sx; x) .$$

Since $M'(t, sx; sx) = sM'(t, sx; x)$ for all $s \geq 0$, we obtain from (3.2.3) using (3.2.4) that

$$\log M(t, sx) \bigg|_{s=1}^{s=2} = \int_1^2 \frac{M'(t, sx; x)}{M(t, sx)} ds < c \int_1^2 \frac{ds}{s} = \log 2c .$$

Therefore, upon simplifying the last inequality, we arrive at

$$M(t, 2x) < 2cM(t, x)$$
whenever $|x| \geq \delta(t)$ proving the first part of the theorem.

The last inequality (3.2.2) in the theorem is obtained from (3.2.5) whenever we integrate over $1 \leq s \leq p, p > 1$.

Inequality (3.2.2) states that $GN$-functions which satisfy a $\Delta$-condition do not grow faster than a power function along any ray passing through the origin. Let us also observe that any function $M(t, x)$ defined on $T \times E^n$ which is either subadditive or a positive homogeneous (of degree one) convex function always satisfies a $\Delta_0$-condition.

4. Generalized mean functions. In this section an integral mean will be defined for $GN$-functions. We will show under what conditions the mean function is a $GN$-function and satisfies a $\Delta$-condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

Let us begin with a definition.

DEFINITION 4.1. For each $t$ in $T$ and $h > 0$ let

$$
M_h(t, x) = \int_{E^n} M(t, x + y)J_h(y)dy
$$

where $J_h(y)$ is a nonnegative, $c^\infty$ function with compact support in a ball of radius $h$ such that $\int_{E^n} J_h(y)dt = 1$. Moreover, let $x_0$ be any point (depending on $h, t$) which satisfies the inequality

$$
M_h(t, x_0) \leq M_h(t, x)
$$

for all $x$ in $E^n$. Then the function $\hat{M}_h(t, x)$ defined for each $t$ in $T$ and $h > 0$ by

$$
\hat{M}_h(t, x) = M_h(t, x + x_0) - M_h(t, x_0)
$$

is called a mean function for $M(t, x)$ relative to the minimizing point $x_0$.

The next theorem shows under what condition $\hat{M}_h(t, x)$ is a $GN$-function.

THEOREM 4.1. If $M(t, x)$ is a $GN$-function for which $\hat{M}(t, c)$ is integrable in $t$ for each $c$, then $\hat{M}_h(t, x)$ is a $GN$-function.

We will show this result by justifying conditions (i)—(iv) of Definition 2.1. By hypothesis and the choice of $x_0$, we have for each $h$, $\hat{M}_h(t, x) \geq 0$ and $\hat{M}_h(t, 0) = 0$. On the other hand, if $x \neq 0$, then
$M(t, x) > 0$, and hence there is constant $h_0$ such that
\[ a = \inf_{|z| \leq h_0} M(t, x + z) > 0. \]
However, since $M(t, x) = 0$ if and only if $x = 0$, the minimizing points $x_0$ tend to zero as $h$ tends to zero. Therefore, we can choose $g_0 \leq h_0$ such that if $h \leq g_0$, then $M(t, x_0 + y) < a$ for all $y$ for which $|x_0 + y| < h$.
For this $g_0$ we obtain the inequality
\[ M(t, x + x_0 + y) \geq \inf_{|z| \leq g_0} M(t, x + z) \geq a > M(t, x_0 + y) \]
whenever $|x_0 + y| \leq g_0$. This means for some $h \leq g_0$ we have
\[ M_h(t, x + x_0) > M_h(t, x_0) \]
or $M_h(t, x) > 0$ if $x \neq 0$ which proves property (i).
Properties (ii) and (iii) for $M_h(t, x)$ follow easily from the same properties for $M(t, x)$. Let us now show (iv). By assumption, there is a constant $d \geq 0$ such that
\[ (4.1.1) \quad l(t) M(t, c) \leq M(t, c) \]
for all $c \geq d$. Furthermore, it is not difficult to show that for all $c$ we have
\[ (4.1.2) \quad \tilde{M}(t, c) \geq \sup_{|x| \leq c} M(t, x) \]
and for some fixed $z$,
\[ (4.1.3) \quad \inf_{|x| \leq c} M(t, x + z) \leq \inf_{|x| = c} M(t, x + z). \]
Using (4.1.2), we obtain for each $t$ in $T$ that
\[ (4.1.4) \quad l(t) \sup_{|z| = c} M(t, z) \leq l(t) \sup_{|w| < c + |x_0 + y|} M(t, w) \leq l(t) \sup_{|w| = c + |x_0 + y|} M(t, w) \]
where $z = x + x_0 + y$. On the other hand, by (4.1.1) and (4.1.3), we achieve
\[ (4.1.5) \quad \inf_{|w| = c + |x_0 + y|} M(t, w) \leq \inf_{|w| = c + |x_0 + y|} M(t, w) < \inf_{|z| = c} M(t, x + x_0 + y) \leq \inf_{|z| = c} M(t, x + x_0 + y). \]
If we combine (4.1.4) and (4.1.5), then for all $c \geq d$ we arrive at
\[ l(t) \sup_{|z| = c} M(t, x + x_0 + y) \leq \inf_{|z| = c} M(t, x + x_0 + y). \]
From this inequality we obtain

\[ \inf_{|x|=c} \mathcal{M}_h(t, x) \geq \int \inf_{|x|=c} \{M(t, x + x_0 + y) - M(t, x_0 + y)\} J_h(y) dy \]

\[ \geq \int \{l(t) \sup_{|x|=c} M(t, x + x_0 + y) - M(t, x_0 + y)\} J_h(y) dy \]

and

\[ \sup_{|x|=c} \mathcal{M}_h(t, x) \leq \int \sup_{|x|=c} M(t, x + x_0 + y) J_h(y) dy . \]

Moreover, since \( \lim_{|y| \to \infty} \sup_{|x|=c} M(t, x + x_0 + y) = \infty \) for fixed \( x_0, y \) such that \( |y| \leq h \), given \( K_1(t) = 2 \sup_{|y| \leq h} M(t, x_0 + y)/\inf l(t) \) there is a \( d_1 > 0 \) such that if \( c \geq d_1 \), then \( \sup_{|x|=c} M(t, x + x_0 + y) \geq K_1 \). Therefore, using (4.1.6) and (4.1.7), we achieve the inequalities

\[ \inf_{|x|=c} \mathcal{M}_h(t, x) \geq l(t) - \frac{1}{2} \inf l(t) \]

for all \( c \geq d_0 = \max \{d, d_1, |x_0|\} \). Taking the infimum of both sides of (4.1.8) over \( t \), shows the first part of property (iv). To show the latter part, assume \( d_0 > 0 \). Then \( \sup_{|x|=d_0} \mathcal{M}_h(t, x) \) is integrable over \( t \) in \( T \) since it is bounded by the integrable function \( \mathcal{M}(t, d_0) \) where \( d_2 = d_0 + |x_0| + h \). This proves property (iv) and the theorem.

In the next theorem we show under what condition \( \mathcal{M}_h(t, x) \) satisfies a \( \Delta \)-condition.

**Theorem 4.2.** If \( M(t, x) \) is a GN-function satisfying a \( \Delta \)-condition and for which \( \mathcal{M}(t, c) \) is integrable in \( t \) for each \( c \), then \( \mathcal{M}_h(t, x) \) satisfies a \( \Delta \)-condition.

It suffices to show that \( M_h(t, x) \) satisfies a \( \Delta \)-condition. For, \( \mathcal{M}_h(t, x) \) is the sum of a constant and a translation of \( M_h(t, x) \) and neither of these operations affects the growth condition. Let us observe first that if \( |x| \geq 2, |y| \leq h \leq 1 \), then \( |2x + y| \leq 3 |x + y| \). Hence, by Theorem 2.2, there are constants \( K \geq 1 \) and \( d_1 \geq 0 \) such that

\[ M_h(t, 2x) \leq K \int \mathcal{M}(t, 3x + y)) J_h(y) dy \]

for all \( x \) such that \( |x| \geq d_2 = \max \{d_1, 2\} \). On the other hand, by Theorem 3.1, there is a constant \( K_2 \geq 2 \) and \( \delta(t) \geq 0 \) such that for almost all \( t \) in \( T \).
for all \( x, y \) such that \( |x + y| \geq \delta(t) \) where \( |y| \leq h \). Combining the above two inequalities we achieve

\[
M_h(t, 2x) \leq KK_M(t, x)
\]

for all \( |x| > \max(d_2, \delta(t) + h) = \delta^*(t) \). Since \( \tilde{M}(t, 2\delta(t)) \) is integrable over \( T \), this yields the integrability of \( \tilde{M}(t, 2\delta(t)) \) proving the theorem.

For each \( t \) in \( T \) and each \( x \) in \( E^* \) it is known that \( \lim_{h \to 0} M_h(t, x) = M(t, x) \). However, the same property does not hold in general for \( \tilde{M}(t, x) \). This is the point of the next theorem.

**Theorem 4.3.** For each \( h > 0 \) let \( x_h^* \) be the minimizing point of \( M_h(t, x) \) defining \( \tilde{M}(t, x) \). Then for each \( t \) in \( T \) and each \( x \) in \( E^* \), there exists \( K(t, x) \) such that

\[
\lim_{h \to 0} \tilde{M}(t, x) = M(t, x) + K(t, x) \lim_{h \to 0} |x_h^*| .
\]

By definition of \( \tilde{M}(t, x) \) we can write

\[
|M(t, x + x_h^* + y) - M(t, x + y) - M(t, x)| J_h(y)dy .
\]

However, we know that

\[
|M(t, x + x_h^* + y) - M(t, x + y) - M(t, x)|
\]

\[
\leq |M(t, x + x_h^* + y) - M(t, x)|
\]

\[
+ |M(t, x_h^* + y) - M(t, y)| + |M(t, y)| .
\]

Moreover, since \( M(t, x) \) is a convex function, it satisfies a Lipshitz condition on compact subsets of \( E^* \) (see, [8, Th. 5.1]). Therefore, there exist \( K_1(t, x) \) and \( K_2(t, x) \) such that

\[
|M(t, x + x_h^* + y) - M(t, x)| \leq K_1(t, x) |x_h^* + y|
\]

and

\[
|M(t, x_h^* + y) - M(t, y)| \leq K_2(t, x) |x_h^*| .
\]

If we combine (4.3.3) and (4.3.4) with (4.3.2) and if we substitute the resulting expression into (4.3.1), we achieve the inequality

\[
|\tilde{M}(t, x) - M(t, x)| \leq |x_h^*| (K_1(t, x) + K_2(t, x))
\]

\[
+ \int_{E^*} K_1(t, x) |y| J_h(y)dy + \int_{E^*} |M(t, y)| J_h(y)dy .
\]
Since the last two integrals on the right side tend to zero as \( h \) tends to zero, we prove the theorem by setting \( K(t, x) = K_1(t, x) + K_2(t, x) \).

**Corollary 4.3.1.** Suppose \( M(t, x) \) is a GN-function such that \( M(t, x) = M(t, -x) \). Then for each \( t \) in \( T \) and \( x \) in \( E^n \),
\[
\lim_{h \to 0} M_h(t, x) = \bar{M}(t, x).
\]

This result is clear since \( \lim_{h \to 0} x_h = 0 \) if \( M(t, x) = M(t, -x) \). In fact, if \( M(t, x) \) is even in \( x \) then the \( x_h = 0 \) for all \( h \).

For each \( t \) in \( T \) let \( A_h \) denote the set of minimizing points of \( M_h(t, x) \) and let \( B \) represent the null space of \( M(t, x) \) relative to points in \( E^n \), i.e.,
\[
B = \{ y \in E^n : M(t, y) = 0 \}.
\]

If \( M(t, x) \) is a GN-function, then \( B = \{0\} \). For the sake of argument, let us suppose that \( M(t, x) \) has all the properties of a GN-function except that \( M(t, x) = 0 \) need not imply \( x = 0 \). We will show the relationships that exist between \( A_h \) and \( B \). This is the content of the next few theorems.

**Theorem 4.4.** The sets \( B \) and \( A_h \) are closed convex sets.

This result follows from the convexity and continuity of \( M(t, x) \) in \( x \) for each \( t \) in \( T \).

**Theorem 4.5.** Let \( B_e = \{ x : M(t, x) < e \} \) for each \( t \) in \( T \). Then given any \( e > 0 \), there is a constant \( h_o > 0 \) such that \( A_h \subseteq B_e \) for each \( h \leq h_o \).

Since \( B \subseteq B_e \), we can choose \( h_o \) sufficiently small so that if \( x \) is in \( B \), then \( x + y \) is in \( B \), for all \( y \) such that \( |y| \leq h_o \). Let \( z \) be an arbitrary but fixed point in \( A_h \), \( h \leq h_o \). Then
\[
M_h(t, z) \leq M_h(t, x)
\]
for all \( x \). Therefore, if \( x \) is in \( B \), we have by our choice of \( h_o \) that \( M_h(t, x) < e \). Letting \( h \) tend to zero yields \( M(t, z) < e \), i.e., \( z \) in \( B_e \).

We have commented above that \( A_h = \{0\} \) if \( M(t, x) = M(t, -x) \). It is also true if \( M(t, x) \) is strictly convex in \( x \) for each \( t \) in \( T \).

**Theorem 4.6.** Suppose \( M(t, x) \) is a GN-function which is strictly convex in \( x \) for each \( t \). Then for each \( h \), \( A_h = \{0\} \).

Suppose there exists \( y_o \neq x_o \) such that \( x_o, y_o \) are in \( A_h \). Let \( z = \)
Then, since $M(t, x)$ is strictly convex, $M_h(t, x)$ is strictly convex in $x$. Therefore, we have

\begin{equation}
M_h(t, z) < \frac{1}{2} M_h(t, x_0) + \frac{1}{2} M_h(t, y_0).
\end{equation}

However, $x_0, y_0$ being in $A_h$ reduces (4.6.1) to the inequality

$$M_h(t, z) < M_h(t, x)$$

for all $x$. This means $z$ is in $A_h$ and $x_0, y_0$ are not in $A_h$ which is a contradiction. Hence, $x_0 = y_0$. Since $M(t, x)$ is a $GN$-function, $B = \{0\}$. In this case $x_0 = y_0 = 0$.

5. Conjugate $GN$-functions. In the study of Orlicz spaces the concept of a conjugate $N$-function plays a significant role. In particular, the definition of these linear spaces may involve a conjugate function. The study of convex functions of several variables and their related conjugate functions can be found in [1, 2, 3, 5].

In this section the concept of a generalized conjugate function is defined and some of its important properties are examined. Many of the standard results which hold for $N$-functions and conjugate functions of a real variable will be generalized here.

We begin with the main definition.

**DEFINITION 5.1.** Let $M(t, x)$ be a $GN$-function. Then we call $M^*(t, x)$ the conjugate function of $M(t, x)$ if for each $t$ in $T$

\begin{equation}
M^*(t, x) = \sup_{z \in E^n} \{zx - M(t, z)\}.
\end{equation}

The notation $zx$ represents the scalar product of the vectors $x$ and $z$.

Let us observe that if $zx \leq 0$ in $(+)$, then $zx - M(t, z) \leq 0$. This means we could, equivalently, restrict the definition to those $z$ for which $zx \geq 0$. Moreover, the equation $(+)$ yields immediately for each $t$ in $T$ that

\begin{equation}
zx \leq M(t, z) + M^*(t, x)
\end{equation}

for all $z, x$ in $E^n$. Inequality $(++)$ could have been used as a definition of the conjugate function.

Fenchel [3] states that to every $z$ in $E^n$ such that $M'(t, z; y) < \infty$ for all $y$ for which it is defined, there is at least one point $x$ in $E^n$ such that equality holds in $(++)$. However, by [8, Th. 5.2] when applied to $GN$-functions, we know for $z$ in $E^n$ that $M'(t, z; y) < \infty$ for all $y$. Therefore, the supremum in $(+)$ is attained for at least one point.
The next theorem gives a necessary and sufficient condition in order that equality hold in $(++)$.

**Theorem 5.1.** Let $M(t, x)$ be a GN-function for which $M'(t, x; y)$ is linear in $y$. Then, given any $x_0, z^i = M'(t, x_0; e_i)$ for all $i = 1, \ldots, n$ if and only if $zx_0 = M(t, x_0) + M^*(t, z)$ where $\{e_i\}$ is a basis for $E^n$.

Clearly, if

$$zx_0 = M(t, x_0) + M^*(t, z)$$

for each $t$ in $T$, then $z^i = M'(t, x_0; e_i)$ for each $i$. On the other hand, suppose $z^i = M'(t, x_0; e_i)$ for each $i = 1, \ldots, n$. Then, by convexity of $M(t, x)$ and linearity of $M'(t, x; y)$, we have for $t$ in $T$

$$(5.1.1)\quad M(t, x) \geq M(t, x_0) + z(x - x_0)$$

for all $x$ in $E^n$. Rewriting (5.1.1) we obtain for all $x$ in $E^n$

$$x_0z - M(t, x_0) \geq xz - M(t, x).$$

Therefore, we have

$$x_0z - M(t, x_0) \geq \sup_x \{xz - M(t, x)\} = M^*(t, z)$$

or

$$(5.1.2)\quad x_0z \geq M(t, x_0) + M^*(t, z).$$

Since $(++)$ always holds, combining (5.1.2) with $(++)$ shows that equality holds in (5.1.2).

The properties of GN-functions possessed by $M^*(t, x)$ are given in the next result.

**Theorem 5.2.** Let $M(t, x)$ be a GN-functions for which

$$\lim_{|x| \to 0} \frac{M(t, x)}{|x|} = 0$$

for each $t$ in $T$. Then $M^*(t, x)$ satisfies properties (i)—(iii) of Definition 2.1. Moreover, if $M(t, x) = M(t, -x)$, then

$$M^*(t, x) = M^*(t, -x).$$

Condition (i) for $M^*(t, x)$ follows directly from the same condition for $M(t, x)$ and the equation in the hypothesis. Convexity follows from the inequality
\[
M^*(t, ax + by) = \sup \{axz - aM(t, z) + byz + bM(t, z)\} \\
\leq aM^*(t, x) + bM^*(t, y)
\]
where \( a + b = 1, a \geq 0, b \geq 0 \). Measurability in \( t \) also follows from the same property for \( M(t, x) \). Finally, if we substitute \( z = kx/|x|, k > 1 \) into \((+++)\) we arrive at

\[
(5.2.1) \quad \frac{M^*(t, x)}{|x|} \geq k - \frac{M(t, kx/|x|)}{|x|}.
\]

However, \( M(t, kx/|x|) \) is bounded on every compact set in \( E^n \) (see [8, Th. 2.5]). Letting \(|x|\) tend to infinity in \((5.2.1)\) results in property (iii).

Suppose \( M(t, x) \) is an even function of \( x \). Then

\[
M^*(t, x) = \sup \{-zx - M(t, -z)\} = \sup \{z(-x) - M(t, z)\} = M^*(t, -x).
\]

Finally, we give conditions when \( M(t, x) \) is the conjugate function of \( M^*(t, x) \).

**Theorem 5.3.** Suppose \( M(t, x) \) is a GN-function for which \( M'(t, x; y) \) is linear in \( y \). Then \( M(t, x) \) is the conjugate function of \( M^*(t, x) \).

Since \( M(t, x) \) is convex in \( x \) and \( M'(t, x; y) \) is linear in \( y \), we achieve for any \( x, x_0 \) in \( E^n \).

\[
M(t, x) - M(t, x_0) \geq M'(t, x_0; x - x_0) \\
\geq M'(t, x_0; x) - M'(t, x_0; x_0)
\]
from which it follows that

\[
(5.3.1) \quad M'(t, x_0; x_0) - M(t, x_0) \leq \sup \{xy - M(t, x)\}
\]
where \( y^i = M'(t, x_0; e_i) \) for each \( i = 1, \ldots, n \) and \( \{e_i\} \) basis vectors for \( E^n \). On the other hand, it is clear that

\[
(5.3.2) \quad M'(t, x_0; x_0) - M(t, x_0) \leq \sup \{xy - M(t, x)\}
\]

since \( M'(t, x_0; x_0) = x_0y \). Combining \((5.3.1)\) and \((5.3.2)\) we obtain the equation

\[
(5.3.3) \quad x_0y - M(t, x_0) = M^*(t, y).
\]

However, by \((++)\), we know that

\[
(5.3.4) \quad xzx \leq M(t, x_0) + M^*(t, z)
\]
for all \( x_0, z \) in \( E^n \). Rewriting (5.3.4) yields

\[
(5.3.5) \quad M(t, x_0) \geq \sup_z \{x_0z - M^*(t, z)\}.
\]

Since (5.3.3) holds for some \( y \), it follows that

\[
(5.3.6) \quad M(t, x_0) = x_0y - M^*(t, y) \leq \sup_z \{x_0z - M^*(t, z)\}.
\]

Therefore, combining (5.3.5) and (5.3.6) produces the desired result that

\[
M(t, x_0) = \sup_z \{x_0z - M^*(t, z)\}.
\]

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