Pacific Journal of Mathematics

VECTOR VALUED ORLICZ SPACES GENERALIZED *N*-FUNCTIONS. I

MICHAEL SAMUEL SKAFF

Vol. 28, No. 1 March 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

VECTOR VALUED ORLICZ SPACES GENERALIZED N-FUNCTIONS, I.

M. S. SKAFF

The theory of Orlicz spaces generated by N-functions of a real variable is well known. On the other hand, as was pointed out by Wang, this same theory generated by N-functions of more than one real variable has not been discussed in the literature. The purpose of this paper is to develop and study such a class of generalized N-functions (called GN-functions) which are a natural generalization of the functions studied by Wang and the variable N-functions by Portnov. In second part of this study we will utilize GN-functions to define vector-valued Orlicz spaces and examine the resulting theory.

This paper is divided into five sections. In § 2, we define and examine some basic properties of GN-functions. A generalized delta condition is introduced and characterized in § 3. In § 4 and § 5 we present, respectively, the theory of an integral mean for GN-functions and the concept of a conjugate GN-function. A complete bibliography on Orlicz spaces, N-functions, and related material can be found in [4, 8]. The study of variable N-functions by Portnov can be found in [6, 7] and the study of nondecreasing N-functions by Wang in [9].

2. GN-functions. In what follows T will denote a space of points with σ-finite measure and E^n n dimensional Euclidean space.

DEFINITION 2.1. Let $M(t, x)$ be a real valued nonnegative function defined on $T \times E^n$ such that

(i) $M(t, x) = 0$ if and only if $x = 0$ for all $t \in T, x \in E^n$,

(ii) $M(t, x)$ is a continuous convex function of x for each t and a measurable function of t for each x,

(iii) For each $t \in T$, $\lim_{|x|=\infty} \dfrac{M(t, x)}{|x|} = \infty$, and

(iv) There is a constant $d \geqq 0$ such that

$$(*) \qquad \inf_t \inf_{c \geqq d} k(t, c) > 0$$

where

$$k(t, c) = \frac{\underline{M}(t, c)}{\bar{M}(t, c)}, \ \bar{M}(t, c) = \sup_{|x|=c} M(t, x) ,$$
$$\underline{M}(t, c) = \inf_{|x|=c} M(t, x)$$

and if $d > 0$, then $\bar{M}(t, d)$ is an integrable function of t. We call a function satisfying properties (i)—(iv) a *generalized N-function* or a *GN-function.*

GN-functions are coordinate independent and are not necessarily symmetric. Therefore, such functions as $M(t, x) = x_1^2 + x_2^2 + (x_1 - x_2)^2$ which are not nondecreasing (as defined in [9]) are allowed in the class of *GN*-functions. The next theorem illustrates this point.

THEOREM 2.1. *If $M(t, x)$ is a GN-function and A is an orthogonal linear transformation defined on E^n with range in E^n, then $\tilde{M}(t, x) = M(t, Ax)$ is a GN-function.*

Properties (i)—(iv) when applied to $\tilde{M}(t, x)$ follow immediately from the same properties for $M(t, x)$ (see [8, Th. 8.1]).

The next theorem characterizes a part of property (iv) in Definition 2.1 and provides a means of comparing function values at different points for *GN*-functions when $|x|$ is large.

THEOREM 2.2. *A necessary and sufficient condition that* (*) *hold is that if $|x| \leqq |y|$, then there exist constants $K \geqq 1$ and $d \geqq 0$ such that $M(t, x) \leqq KM(t, y)$ for each $t \in T$ and $|x| \geqq d$.*

If (*) is true, then there exists a constant $d \geqq 0$ such that $l(t) = \inf_{c \geqq d} k(t, c) > 0$ for each t in T. By definition of $k(t, c)$ this means

$$M(t, y) \geqq \underline{M}(t, |y|) \geqq l(t)\bar{M}(t, |y|) \tag{2.2.1}$$

for any y such that $|y| = c \geqq d$. On the other hand, if $d \leqq |x| \leqq |y|$, then the convexity of $M(t, x)$ and $M(t, 0) = 0$ yields

$$\bar{M}(t, |y|) \geqq \sup_{|z|=|x|} M(t, z) . \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2) we arrive at

$$M(t, y) \geqq l(t) \sup_{|z|=|x|} M(t, z) \geqq K^{-1}M(t, x)$$

whenever $d \leqq |x| \leqq |y|$ where $K^{-1} = \inf_t l(t) > 0$

The converse follows easily from the condition in the theorem.

It is interesting to note that if $M(t, x)$ is a *GN*-function, then $2\hat{M}(t, x) = M(t, x) + \tilde{M}(t, x)$ is also a *GN*-function where $\tilde{M}(t, x)$ is defined as in Theorem 2.1. This means we can construct a symmetric (in x) *GN*-function from one which does not possess this property. For, if $\tilde{M}(t, x) = M(t, -x)$, then $\hat{M}(t, x)$ is clearly symmetric in x.

Property (iv) of Definition 2.1 provides the condition which allows a natural generalization from N-functions of a real variable to those of several real variables. Let us observe that the function $\bar{M}(t, c)$ is also a GN-function of a real nonnegative variable c. On the other hand, $M(t, c)$ need not even be convex in c.

Since $\underline{M}(t, c) \leqq M(t, x) \leqq \bar{M}(t, c)$ for each x such that $|x| = c$, we would like to find a GN-function which bounds $\underline{M}(t, c)$ from below for all c. If $d = 0$ in Theorem 2.2, then $K^{-1}\bar{M}(t, c)$ would do.

One might accomplish the construction of such a function by taking the supremum of a class of convex functions bounding $\underline{M}(t, c)$ from below. This function would be convex. However, this class may be empty. The next theorem shows that this is not the case whenever $M(t, x)$ is a GN-function. The construction employed can be applied to more general settings than exist here.

THEOREM 2.3. *If $M(t, x)$ is a GN-function and $\underline{M}(t, c)$ is defined as above, then there exists a GN-function $R(t, c)$ such that $R(t, c) \leqq \underline{M}(t, c)$ for all $c \geqq 0$.*

Since $\underline{M}(t, c)$ satisfies property (iii) of Definition 2.1, given any $d > 0$ there is a $c_0 > 0$ such that $\underline{M}(t, c) \geqq dc$ whenever $c \geqq c_0$. Let us define the function

$$P(t, c) = \begin{cases} \sup\limits_{\substack{0 < w \leqq 1 \\ cw \geqq c_0}} \dfrac{\underline{M}(t, cw)}{w} & \text{if } c \geqq c_0 \\ \underline{M}(t, c) & \text{if } 0 \leqq c < c_0 \,. \end{cases}$$

Then it is easy to show that (i) $P(t, ac) \leqq aP(t, c)$ for $0 \leqq a \leqq 1$, (ii) $\{P(t, c)/c\}$ is a nondecreasing function of c, and (iii) $P(t, c)$ is finite for each c. We now obtain the desired function $R(t, c)$ by defining

$$R(t, c) = \int_0^c Q(t, s)ds$$

where

$$Q(t, c) = \begin{cases} \dfrac{P(t, c)}{c} & \text{if } c \geqq c_0 \\ \dfrac{cP(t, c_0)}{c_0^2} & \text{if } 0 \leqq c < c_0 \,. \end{cases}$$

We have immediately that

$$R(t, c) \leqq cQ(t, c) = P(t, c) \leqq \underline{M}(t, c) \,.$$

If is not difficult to show that $R(t, c)$ is also a GN-function.

3. Delta condition. In this section a generalized growth condition is defined for GN-functions. This growth or delta condition generalizes that definition usually given for a real variable N-function (e.g., see [4, 6, 7]).

DEFINITION 3.1. We say a GN-function $M(t, x)$ satisfies a *Δ-condition* if there exist a constant $K \geqq 2$ and a non-negative measurable function $\delta(t)$ such that the function $\bar{M}(t, 2\delta(t))$ is integrable over the domain T and such that for almost all t in T we have

$$(**) \qquad M(t, 2x) \leqq KM(t, x)$$

for all x satisfying $|x| \geqq \delta(t)$.

We say a GN-function satisfies a *Δ_0-condition* if it satisfies a Δ-condition with $\delta(t) = 0$ for almost all t in T.

In Definition 3.1 we could have used any constant $l > 1$ in place of the scalar 2 in (**). This is the basis of the next theorem which gives an equivalent definition to that employed in 3.1.

THEOREM 3.1. *A GN-function $M(t, x)$ satisfies a Δ-condition if and only if given any $l > 1$ there exists a constant $K_l \geqq 2$ and a nonnegative measurable function $\delta(t)$ such that $\bar{M}(t, 2\delta(t))$ is integrable over T and such that for almost all t in T we have*

$$(3.1.1) \qquad M(t, lx) \leqq K_l M(t, x)$$

whenever $|x| \geqq \delta(t)$.

Suppose $M(t, x)$ satisfies a Δ-condition and $l > 1$. We choose m so large that $2^m \geqq l$. Then by convexity and our assumption of a Δ-condition there is a $K \geqq 2$ and measurable $\delta(t) \geqq 0$ such that for almost all t in T

$$M(t, lx) \leqq M(t, 2^m x) \leqq K^m M(t, x)$$

whenever $|x| \geqq \delta(t)$. Therefore (3.1.1) holds with $K_l = K^m$. The converse follows as easily.

Whenever we deal with convex functions of several variables the concept of a one sided directional derivative plays an important role. The next result utilizes such a function, so we define it now.

DEFINITION 3.2. For each t in T the *directional derivative of a GN-function $M(t, x)$ in a direction y* is defined by

$$M'(t, x; y) = \lim_{h=0+} \frac{M(t, x + hy) - M(t, x)}{h} .$$

The important properties of directional derivatives of convex functions of several variables which will be needed can be found in [3, 8]. Using the directional derivative defined above, the next result characterizes the delta condition and generalizes similar results given in [4, 6, 7].

THEOREM 3.2. *A GN-function $M(t, x)$ satisfies a Δ-condition if and only if there exists a nonnegative measurable function $\delta(t)$ such that $\bar{M}(t, 2\delta(t))$ is integrable over T and a constant $c > 1$ such that for almost all t in T*

$$\frac{M'(t, x; x)}{M(t, x)} < c \tag{3.2.1}$$

whenever $|x| \geqq \delta(t)$. Moreover, if (3.2.1) holds, then for almost all t in T and for each x such that $|x| \geqq \delta(t)$ we have

$$M(t, px) < M(t, x)p^c \tag{3.2.2}$$

for all $p > 1$.

Suppose $M(t, x)$ satisfies a Δ-condition. Then, by convexity (see, [8, Th. 5.3]), we must have for some $K \geqq 2$ and $\delta(t) \geqq 0$

$$KM(t, x) \geqq M(t, 2x) \geqq M(t, x) + M'(t, x; x)$$

whenever $|x| \geqq \delta(t)$. This means (3.2.1) holds with $c = K$.

Conversely, suppose (3.2.1) holds. We choose s such that $s \geqq 1$. Then, by assumption, there is a constant $c > 1$ and $\delta(t) > 0$ such that for almost all t in T

$$\frac{M'(t, sx; sx)}{M(t, sx)} > c \tag{3.2.3}$$

whenever $|x| \geqq \delta(t)$. On the other hand, we have

$$\begin{aligned} \frac{d}{ds} M(t, sx) &= \lim_{h=0^+} \frac{M(t, sx + hx) - M(t, sx)}{h} \\ &= M'(t, sx; x) . \end{aligned} \tag{3.2.4}$$

Since $M'(t, sx; sx) = sM'(t, sx; x)$ for all $s \geqq 0$, we obtain from (3.2.3) using (3.2.4) that

$$\log M(t, sx) \Big|_{s=1}^{s=2} = \int_1^2 \frac{M'(t, sx; x)}{M(t, sx)} ds < c \int_1^2 \frac{ds}{s} = \log 2^c . \tag{3.2.5}$$

Therefore, upon simplifying the last inequality, we arrive at

$$M(t, 2x) < 2^c M(t, x)$$

whenever $|x| \geqq \delta(t)$ proving the first part of the theorem.

The last inequality (3.2.2) in the theorem is obtained from (3.2.5) whenever we integrate over $1 \leqq s \leqq p$, $p > 1$.

Inequality (3.2.2) states that GN-functions which satisfy a Δ-condition do not grow faster than a power function along any ray passing through the origin. Let us also observe that any function $M(t, x)$ defined on $T \times E^n$ which is either subadditive or a positive homogeneous (of degree one) convex function always satisfies a Δ_0-condition.

4. Generalized mean functions. In this section an integral mean will be defined for GN-functions. We will show under what conditions the mean function is a GN-function and satisfies a Δ-condition. Moreover, we examine how the minimizing points in the definition of the mean function affect a basic property of the ordinary integral mean.

Let us begin with a definition.

DEFINITION 4.1. For each t in T and $h > 0$ let

$$M_h(t, x) = \int_{E^n} M(t, x + y) J_h(y) dy$$

where $J_h(y)$ is a nonnegative, c^∞ function with compact support in a ball of radius h such that $\int_{E^n} J_h(y) dt = 1$. Moreover, let x_0 be any point (depending on h, t) which satisfies the inequality

$$M_h(t, x_0) \leqq M_h(t, x)$$

for all x in E^n. Then the function $\hat{M}_h(t, x)$ defined for each t in T and $h > 0$ by

$$\hat{M}_h(t, x) = M_h(t, x + x_0) - M_h(t, x_0)$$

is called a *mean function for $M(t, x)$ relative to the minimizing point x_0*.

The next theorem shows under what condition $\hat{M}_h(t, x)$ is a GN-function.

THEOREM 4.1. *If $M(t, x)$ is a GN-function for which $\bar{M}(t, c)$ is integrable in t for each c, then $\hat{M}_h(t, x)$ is a GN-function.*

We will show this result by justifying conditions (i)—(iv) of Definition 2.1. By hypothesis and the choice of x_0, we have for each h, $\hat{M}_h(t, x) \geqq 0$ and $\hat{M}_h(t, 0) = 0$. On the other hand, if $x \neq 0$, then

$M(t, x) > 0$, and hence there is constant h_0 such that

$$a = \inf_{|z| \leqq h_0} M(t, x + z) > 0 .$$

However, since $M(t, x) = 0$ if and only if $x = 0$, the minimizing points x_0 tend to zero as h tends to zero. Therefore, we can choose $g_0 \leqq h_0$ such that if $h \leqq g_0$, then $M(t, x_0 + y) < a$ for all y for which $|x_0 + y| < h$. For this g_0 we obtain the inequality

$$M(t, x + x_0 + y) \geqq \inf_{|z| \leqq g_0} M(t, x + z) \geqq a > M(t, x_0 + y)$$

whenever $|x_0 + y| \leqq g_0$. This means for some $h \leqq g_0$ we have

$$M_h(t, x + x_0) > M_h(t, x_0)$$

or $\hat{M}_h(t, x) > 0$ if $x \neq 0$ which proves property (i).

Properties (ii) and (iii) for $\hat{M}_h(t, x)$ follow easily from the same properties for $M(t, x)$. Let us now show (iv). By assumption, there is a constant $d \geqq 0$ such that

$$l(t)\bar{M}(t, c) \leqq \underline{M}(t, c) \tag{4.1.1}$$

for all $c \geqq d$. Furthermore, it is not difficult to show that for all c we have

$$\bar{M}(t, c) \geqq \sup_{|x| \leqq c} M(t, x) \tag{4.1.2}$$

and for some fixed z,

$$\inf_{|x| \geqq c} M(t, x + z) \leqq \inf_{|x| = c} M(t, x + z) . \tag{4.1.3}$$

Using (4.1.2), we obtain for each t in T that

$$\begin{aligned} l(t) \sup_{|x| = c} M(t, z) &\leqq l(t) \sup_{|w| < c + |x_0 + y_1|} M(t, w) \\ &\leqq l(t) \sup_{|w| = c + |x_0 + y_1|} M(t, w) \end{aligned} \tag{4.1.4}$$

where $z = x + x_0 + y$. On the other hand, by (4.1.1) and (4.1.3), we achieve

$$\begin{aligned} l(t) \sup_{|w| = c + |x_0 + y_1|} M(t, w) &\leqq \inf_{|w| = c + |x_0 + y_1|} M(t, w) \\ &< \inf_{|x| \geqq c} M(t, x + x_0 + y) \\ &< \inf_{|x| = c} M(t, x + x_0 + y) . \end{aligned} \tag{4.1.5}$$

If we combine (4.1.4) and (4.1.5), then for all $c \geqq d$ we arrive at

$$l(t) \sup_{|x| = c} M(t, x + x_0 + y) \leqq \inf_{|x| = c} M(t, x + x_0 + y) .$$

From this inequality we obtain

$$\inf_{|x|=c} \hat{M}_h(t, x) \geqq \int_{E^n} \inf_{|x|=c} \{M(t, x + x_0 + y) - M(t, x_0 + y)\} J_h(y) dy \tag{4.1.6}$$
$$\geqq \int_{E^n} \{l(t) \sup_{|x|=c} M(t, x + x_0 + y) - M(t, x_0 + y)\} J_h(y) dy$$

and

$$\sup_{|x|=c} \hat{M}_h(t, x) \leqq \int_{E^n} \sup_{|x|=c} M(t, x + x_0 + y) J_h(y) dy\ . \tag{4.1.7}$$

Moreover, since $\lim_{c=\infty} \sup_{|x|=c} M(t, x + x_0 + y) = \infty$ for fixed x_0, y such that $|y| \leqq h$, given $K_1(t) = 2 \sup_{|y|\leqq h} M(t, x_0 + y)/\inf_t l(t)$ there is a $d_1 > 0$ such that if $c \geqq d_1$, then $\sup_{|x|=c} M(t, x + x_0 + y) \geqq K_1$. Therefore, using (4.1.6) and (4.1.7), we achieve the inequalities

$$\frac{\inf_{|x|=c} \hat{M}_h(t, x)}{\sup_{|x|=c} \hat{M}_h(t, x)} \geqq l(t) - \frac{\sup_{|y|\leqq h} M(t, x_0 + y)}{\inf_{|y|\leqq h} \sup_{|x|=c} M(t, x + x_0 + y)} \tag{4.1.8}$$
$$\geqq l(t) - \frac{1}{2} \inf_t l(t)$$

for all $c \geqq d_0 = \max(d, d_1, |x_0|)$. Taking the infimum of both sides of (4.1.8) over t, shows the first part of property (iv). To show the latter part, assume $d_0 > 0$. Then $\sup_{|x|=d_0} \hat{M}_h(t, x)$ is integrable over t in T since it is bounded by the integrable function $\bar{M}(t, d_2)$ where $d_2 = d_0 + |x_0| + h$. This proves property (iv) and the theorem.

In the next theorem we show under what condition $\hat{M}_h(t, x)$ satisfies a Δ-condition.

THEOREM 4.2. *If $M(t, x)$ is a GN-function satisfying a Δ-condition and for which $\bar{M}(t, c)$ is integrable in t for each c, then $\hat{M}_h(t, x)$ satisfies a Δ-condition.*

It suffices to show that $M_h(t, x)$ satisfies a Δ-condition. For, $\hat{M}_h(t, x)$ is the sum of a constant and a translation of $M_h(t, x)$ and neither of these operations affects the growth condition. Let us observe first that if $|x| \geqq 2, |y| \leqq h \leqq 1$, then $|2x + y| \leqq 3|x + y|$. Hence, by Theorem 2.2, there are constants $K \geqq 1$ and $d_1 \geqq 0$ such that

$$M_h(t, 2x) \leqq K \int_{E^n} M(t, 3(x + y)) J_h(y) dy$$

for all x such that $|x| \geqq d_2 = \max(d_1, 2)$. On the other hand, by Theorem 3.1, there is a constant $K_3 \geqq 2$ and $\delta(t) \geqq 0$ such that for almost all t in T

$$\int_{E^n} M(t, 3(x+y))J_h(y)dy \leqq K_3 M_h(t, x)$$

for all x, y such that $|x+y| \geqq \delta(t)$ where $|y| \leqq h$. Combining the above two inequalities we achieve

$$M_h(t, 2x) \leqq KK_3 M_h(t, x)$$

for all $|x| > \max(d_2, \delta(t) + h) = \delta_1(t)$. Since $\bar{M}(t, 2\delta_1(t))$ is integrable over T, this yields the integrability of $\bar{M}_h(t, 2\delta_1(t))$ proving the theorem.

For each t in T and each x in E^n it is known that $\lim_{h=0} M_h(t, x) = M(t, x)$. However, the same property does not hold in general for $\hat{M}_h(t, x)$. This is the point of the next theorem.

THEOREM 4.3. *For each $h > 0$ let x_0^h be the minimizing point of $M_h(t, x)$ defining $\hat{M}_h(t, x)$. Then for each t in T and each x in E^n, there exists $K(t, x)$ such that*

$$\lim_{h=0} \hat{M}_h(t, x) = M(t, x) + K(t, x) \lim_{h=0} |x_0^h| .$$

By definition of $\hat{M}_h(t, x)$ we can write

$$\begin{aligned} &|\hat{M}_h(t, x) - M(t, x)| \\ &\leqq \int_{E^n} |M(t, x + x_0^h + y) - M(t, x_0^h + y) - M(t, x)| J_h(y)dy . \end{aligned} \tag{4.3.1}$$

However, we know that

$$\begin{aligned} &|M(t, x + x_0^h + y) - M(t, x_0^h + y) - M(t, x)| \\ &\leqq |M(t, x + x_0^h + y) - M(t, x)| \\ &\quad + |M(t, x_0^h + y) - M(t, y)| + |M(t, y)| . \end{aligned} \tag{4.3.2}$$

Moreover, since $M(t, x)$ is a convex function, it satisfies a Lipshitz condition on compact subsets of E^n (see, [8, Th. 5.1]). Therefore, there exist $K_1(t, x)$ and $K_2(t, x)$ such that

$$|M(t, x + x_0^h + y) - M(t, x)| \leqq K_1(t, x) |x_0^h + y| \tag{4.3.3}$$

and

$$|M(t, x_0^h + y) - M(t, y)| \leqq K_2(t, x) |x_0^h| . \tag{4.3.4}$$

If we combine (4.3.3) and (4.3.4) with (4.3.2) and if we substitute the resulting expression into (4.3.1), we achieve the inequality

$$\begin{aligned} |\hat{M}_h(t, x) - M(t, x)| &\leqq |x_0^h| (K_1(t, x) + K_2(t, x)) \\ &+ \int_{E^n} K_1(t, x) |y| J_h(y)dy + \int_{E^n} |M(t, y)| J_h(y)dy . \end{aligned}$$

Since the last two integrals on the right side tend to zero as h tends to zero, we prove the theorem by setting $K(t, x) = K_1(t, x) + K_2(t, x)$.

COROLLARY 4.3.1. *Suppose $M(t, x)$ is a GN-function such that $M(t, x) = M(t, -x)$. Then for each t in T and x in E^n,*

$$\lim_{h=0} M_h(t, x) = \hat{M}(t, x) .$$

This result is clear since $\lim_{h=0} |x_0^h| = 0$ if $M(t, x) = M(t, -x)$. In fact, if $M(t, x)$ is even in x then the $x_0^h = 0$ for all h.

For each t in T let A_h denote the set of minimizing points of $M_h(t, x)$ and let B represent the null space of $M(t, x)$ relative to points in E^n, i.e.,

$$B = \{y \text{ in } E^n: M(t, y) = 0\} .$$

If $M(t, x)$ is a GN-function, then $B = \{0\}$. For the sake of argument, let us suppose that $M(t, x)$ has all the properties of a GN-function except that $M(t, x) = 0$ need not imply $x = 0$. We will show the relationships that exist between A_h and B. This is the content of the next few theorems.

THEOREM 4.4. *The sets B and A_h are closed convex sets.*

This result follows from the convexity and continuity of $M(t, x)$ in x for each t in T.

THEOREM 4.5. *Let $B_e = \{x: M(t, x) < e\}$ for each t in T. Then given any $e > 0$, there is a constant $h_0 > 0$ such that $A_h \subseteqq B_e$ for each $h \leqq h_0$.*

Since $B \subseteqq B_e$, we can choose h_0 sufficiently small so that if x is in B, then $x + y$ is in B_e for all y such that $|y| \leqq h_0$. Let z be an arbitrary but fixed point in A_h, $h \leqq h_0$. Then

$$M_h(t, z) \leqq M_h(t, x)$$

for all x. Therefore, if x is in B, we have by our choice of h_0 that $M_h(t, z) < e$. Letting h tend to zero yields $M(t, z) < e$, i.e., z in B_e.

We have commented above that $A_h = \{0\}$ if $M(t, x) = M(t, -x)$. It is also true if $M(t, x)$ is strictly convex in x for each t in T.

THEOREM 4.6. *Suppose $M(t, x)$ is a GN-function which is strictly convex in x for each t. Then for each h, $A_h = \{0\}$.*

Suppose there exists $y_0 \neq x_0$ such that x_0, y_0 are in A_h. Let $z =$

$(x_0 + y_0)/2$. Then, since $M(t, x)$ is strictly convex, $M_h(t, x)$ is strictly convex in x. Therefore, we have

$$M_h(t, z) < \frac{1}{2} M_h(t, x_0) + \frac{1}{2} M_h(t, y_0) . \tag{4.6.1}$$

However, x_0, y_0 being in A_h reduces (4.6.1) to the inequality

$$M_h(t, z) < M_h(t, x)$$

for all x. This means z is in A_h and x_0, y_0 are not in A_h which is a contradiction. Hence, $x_0 = y_0$. Since $M(t, x)$ is a GN-function, $B = \{0\}$. In this case $x_0 = y_0 = 0$.

5. Conjugate GN-functions. In the study of Orlicz spaces the concept of a conjugate N-fuction plays a significant role. In particular, the definition of these linear spaces may involve a conjugate function. The study of convex functions of several variables and their related conjugate functions can be found in [1, 2, 3, 5].

In this section the concept of a generalized conjugate function is defined and some of its important properties are examined. Many of the standard results which hold for N-functions and conjugate functions of a real variable will be generalized here.

We begin with the main definition.

DEFINITION 5.1. Let $M(t, x)$ be a GN-function. Then we call $M^*(t, x)$ the *conjugate function* of $M(t, x)$ if for each t in T

$$M^*(t, x) = \sup_{z \text{ in } E^n} \{zx - M(t, z)\} . \tag{+}$$

The notation zx represents the scalar product of the vectors x and z.

Let us observe that if $zx \leqq 0$ in (+), then $zx - M(t, z) \leqq 0$. This means we could, equivalently, restrict the definition to those z for which $zx \geqq 0$. Moreover, the equation (+) yields immediately for each t in T that

$$zx \leqq M(t, z) + M^*(t, x) \tag{++}$$

for all z, x in E^n. Inequality (++) could have been used as a definition of the conjugate function.

Fenchel [3] states that to every z in E^n such that $M'(t, z; y) < \infty$ for all y for which it is defined, there is at least one point x in E^n such that equality holds in (++). However, by [8, Th. 5.2] when applied to GN-functions, we know for z in E^n that $M'(t, z; y) < \infty$ for all y. Therefore, the supremum in (+) is attained for at least one point.

The next theorem gives a necessary and sufficient condition in order that equality hold in $(++)$.

THEOREM 5.1. *Let $M(t, x)$ be a GN-function for which $M'(t, x; y)$ is linear in y. Then, given any x_0, $z^i = M'(t, x_0; e_i)$ for all $i = 1, \cdots, n$ if and only if $zx_0 = M(t, x_0) + M^*(t, z)$ where $\{e_i\}$ is a basis for E^n.*

Clearly, if

$$zx_0 = M(t, x_0) + M^*(t, z)$$

for each t in T, then $z^i = M'(t, x_0; e_i)$ for each i. On the other hand, suppose $z^i = M'(t, x_0; e_i)$ for each $i = 1, \cdots, n$. Then, by convexity of $M(t, x)$ and linearity of $M'(t, x; y)$, we have for t in T

$$M(t, x) \geqq M(t, x_0) + z(x - x_0) \tag{5.1.1}$$

for all x in E^n. Rewriting (5.1.1) we obtain for all x in E^n

$$x_0 z - M(t, x_0) \geqq xz - M(t, x) .$$

Therefore, we have

$$x_0 z - M(t, x_0) \geqq \sup_x \{xz - M(t, x)\} = M^*(t, z)$$

or

$$x_0 z \geqq M(t, x_0) + M^*(t, z) . \tag{5.1.2}$$

Since $(++)$ always holds, combining (5.1.2) with $(++)$ shows that equality holds in (5.1.2).

The properties of GN-functions possessed by $M^*(t, x)$ are give in the next result.

THEOREM 5.2. *Let $M(t, x)$ be a GN-functions for which*

$$\lim_{|x|=0} \frac{M(t, x)}{|x|} = 0$$

for each t in T. Then $M^*(t, x)$ satisfies properties (i)—(iii) *of Definition* 2.1. *Moreover, if $M(t, x) = M(t, -x)$, then*

$$M^*(t, x) = M^*(t, -x) .$$

Condition (i) for $M^*(t, x)$ follows directly from the same condition for $M(t, x)$ and the equation in the hypothesis. Convexity follows from the inequality

$$M^*(t, ax + by) = \sup \{axz - aM(t, z) + byz + bM(t, z)\}$$
$$\leqq aM^*(t, x) + bM^*(t, y)$$

where $a + b = 1$, $a \geqq 0$, $b \geqq 0$. Measurability in t also follows from the same property for $M(t, x)$. Finally, if we substitute $z = kx/|x|$, $k>1$ into $(++)$ we arrive at

(5.2.1) $$\frac{M^*(t, x)}{|x|} \geqq k - \frac{M\left(t, \frac{kx}{|x|}\right)}{|x|} .$$

However, $M(t, kx/|x|)$ is bounded on every compact set in E^n (see [8, Th. 2.5]). Letting $|x|$ tend to infinity in (5.2.1) results in property (iii).

Suppose $M(t, x)$ is an even function of x. Then

$$M^*(t, x) = \sup_z \{-zx - M(t, -z)\}$$
$$= \sup_z \{z(-x) - M(t, z)\} = M^*(t, -x) .$$

Finally, we give conditions when $M(t, x)$ is the conjugate function of $M^*(t, x)$.

THEOREM 5.3. *Suppose $M(t, x)$ is a GN-function for which $M'(t, x; y)$ is linear in y. Then $M(t, x)$ is the conjugate function of $M^*(t, x)$.*

Since $M(t, x)$ is convex in x and $M'(t, x; y)$ is linear in y, we achieve for any x, x_0 in E^n.

$$M(t, x) - M(t, x_0) \geqq M'(t, x_0; x - x_0)$$
$$\geqq M'(t, x_0; x) - M'(t, x_0: x_0)$$

from which it follows that

(5.3.1) $$M'(t, x_0; x_0) - M(t, x_0) \geqq \sup_x \{xy - M(t, x)\}$$

where $y^i = M'(t, x_0; e_i)$ for each $i = 1, \cdots, n$ and $\{e_i\}$ basis vectors for E^n. On the other hand, it is clear that

(5.3.2) $$M'(t, x_0; x_0) - M(t, x_0) \leqq \sup_x \{xy - M(t, x)\}$$

since $M'(t, x_0; x_0) = x_0 y$. Combining (5.3.1) and (5.3.2) we obtain the equation

(5.3.3) $$x_0 y - M(t, x_0) = M^*(t, y) .$$

However, by $(++)$, we know that

(5.3.4) $$x_0 z \leqq M(t, x_0) + M^*(t, z)$$

for all x_0, z in E^n. Rewriting (5.3.4) yields

$$M(t, x_0) \geqq \sup_z \{x_0 z - M^*(t, z)\} . \tag{5.3.5}$$

Since (5.3.3) holds for some y, it follows that

$$M(t, x_0) = x_0 y - M^*(t, y) \leqq \sup_z \{x_0 z - M^*(t, z)\} . \tag{5.3.6}$$

Therefore, combining (5.3.5) and (5.3.6) produces the desired result that

$$M(t, x_0) = \sup_z \{x_0 z - M^*(t, z)\} .$$

References

1. A. Brøndsted, *Conjugate convex functions in topological vector spaces,* Mat. Fys. Medd. Dansk. Vid. Silsk. **34** (1964), 3-27.
2. W. Fenchel, *On conjugate convex functions,* Canad. J. Math. **1** (1949), 73-77.
3. ———, *Convex cones, sets, and functions,* Lecture notes, Princeton Univ., 1953.
4. M. A. Krasnoselskii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces* (translation), Noordhoft, Ltd., Groningen, 1961.
5. W. A. J. Luxemburg and A. C. Zaanen, *Conjugate spaces of Orlicz spaces,* Indag. Math. **18** (1956), 217-228.
6. V. R. Portnov, *A contribution to the theory of Orlicz spaces generated by variable N-functions,* Soviet Math. Dokl. **8** (1967), 857-860.
7. ———, *Some properties of Orlicz spaces generated by M(x, w) functions,* Soviet Math. Dokl. **7** (1966), 1377-1380.
8. M. S. Skaff, *Vector valued Orlicz spaces,* Thesis, University of California, Los Angeles, 1968.
9. Wang Sheng-Wang, *Convex functions of several variables and vectorvalued Orlicz spaces,* Bull. Acad. Polon. Sci. Sér. Math. Astr. et Phys. **11** (1963), 279-284.

Received July 10, 1968. The preparation of this paper was sponsored by the U. S. Army Research Office under Grant DA-31-124-ARO(D)-355. Reproduction in whole or in part is permitted for any purpose of the United States Government.

University of California, Los Angeles

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 28, No. 1 March, 1969