Pacific Journal of Mathematics

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Vol. 28, No. 1 March 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

ON H-EQUIVALENCE OF UNIFORMITIES (II)

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This paper, continuing previous work by the same author, is concerned with the following problem: Given a metrisable uniformity $\mathfrak{U}$ for a set X, does there exist another (distinct) uniformity $\mathfrak{V}$ for X such that the two corresponding Hausdorff uniformities induce the same topology on the set, $S(X)$ say, of all nonempty subsets of X? Sufficient conditions for the existence, and sufficient conditions for the nonexistence, of such a uniformity $\mathfrak{V}$ are given, together with related results concerning the Hausdorff uniformities (derived from $\mathfrak{U}$ and $\mathfrak{V}$) for $S(X_1)$, where X_1 is a subset of X, everywhere dense in the topology derived from $\mathfrak{U}$.

The notation is that used in the previous paper [4]; Theorem 1 of that paper will be referred to as Theorem 1A, and so on. We shall also say for brevity that a uniformity $\mathfrak{V}$ is *H-singular* (over X) if and only if there exists no distinct uniformity for X which is H-equivalent to $\mathfrak{V}$ on X.

1. *H*-equivalence on dense subsets.

Our first theorem will allow an improvement of Theorem 4A.

THEOREM 1. *Let $\mathfrak{V}$ be a metrisable uniformity for X (that is, one with an enumerable base in $X \times X$) and X_1 a subset dense in X, in the topology $\mathscr{T}(\mathfrak{V})$ induced by $\mathfrak{V}$. Let $\mathfrak{U}$ be another uniformity for X, such that*

(a) $\mathscr{T}(\mathfrak{U}) \subset \mathscr{T}(\mathfrak{V})$ on X;

(b) *the restrictions $\mathfrak{U}_1$, $\mathfrak{V}_1$ of $\mathfrak{U}$, $\mathfrak{V}$ to $X_1 \times X_1$ are H-equivalent on X_1.*

Then if $\mathfrak{U}$ and $\mathfrak{V}$ are not H-equivalent on X the cardinal of X must be measurable.

We achieve the proof by five propositions, the first two of which do not depend on the metrisability of $\mathfrak{V}$.

(i) $\mathfrak{U} \subset \mathfrak{V}$.

By Theorem 1A[1], $\mathfrak{U}_1$ and $\mathfrak{V}_1$ are proximity-equivalent (on X_1); as $\mathfrak{V}_1$ is metrisable this implies $\mathfrak{U}_1 \subset \mathfrak{V}_1$. Given $U_0 \in \mathfrak{U}$, take a symmetric $U \in \mathfrak{U}$ such that $\overset{3}{U} \subset U_0$, and a symmetric $V \in \mathfrak{V}$ such that $\overset{3}{V} \cap (X_1 \times X_1)$

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[1] The part of Theorem 1A actually used here was proved earlier by D. H. Smith, [**1**, Th. 1].

$\subset U$. Given any $x \in X$, since $\mathscr{T}(\mathfrak{U}) \subset \mathscr{T}(\mathfrak{V})$ and X_1 is dense in X, we have $V(x) \cap U(x) \cap X_1 \neq \varnothing$. Thus if $(x, x') \in V$ there exist x_1, x_1' in X_1 with (x, x_1) and (x', x_1') both in $V \cap U$. Then

$$(x_1, x_1') \in \overset{3}{V} \cap (X_1 \times X_1) \subset U$$

so that $(x, x') \in \overset{3}{U} \subset U_0$. That is, $V \subset U_0$ so that $U_0 \in \mathfrak{V}$.

(ii) $\mathfrak{U}, \mathfrak{V}$ are proximity-equivalent on X; hence $\mathscr{T}(\mathfrak{U}) = \mathscr{T}(\mathfrak{V})$.

Let A, B be $\mathfrak{V}$-remote, say $\overset{3}{V}$-remote where $V \in \mathfrak{V}$ is symmetric. Then $A_1 = V(A) \cap X_1$ and $B_1 = V(B) \cap X_1$ are V-remote subsets of X_1, so (again since $\mathfrak{U}_1, \mathfrak{V}_1$ are proximity-equivalent) there exists symmetric $U \in \mathfrak{U}$ with A_1, B_1 $\overset{3}{U}$-remote. Then $U(A_1), U(B_1)$ are U-remote in X, but as X_1 is dense we have $A \subset (\bar{A}_1; \mathfrak{V}) \subset (\bar{A}_1; \mathfrak{U}) \subset U(A_1)$, where $(\bar{A}_1; \mathfrak{V})$ and $(\bar{A}_1, \mathfrak{U})$ are the closures of A_1 in $\mathscr{T}(\mathfrak{V}), \mathscr{T}(\mathfrak{U})$ respectively. Similarly $B \subset U(B_1)$, so that A, B are also $\mathfrak{U}$-remote; the reverse implication follows at once from (i).

From now on we suppose $\mathfrak{U}, \mathfrak{V}$ not H-equivalent on X. It follows from (i), (ii) and Theorem 1A that there exists a set $E_0 \subset X$ which is $\mathfrak{V}$-discrete but not $\mathfrak{U}$-discrete.

(iii) If $\{E_n; n = 1, 2, \cdots\}$ is a sequence of disjoint subsets of E_0 then, for some N, $U(E_n; n \geqq N)$ is $\mathfrak{U}$-discrete.

We can choose a base $\{V_n; n = 1, 2, \cdots\}$ of $\mathfrak{V}$ such that each V_n is symmetric, $V_{n+1} \subset V_n$ for all n, and E_0 is $\overset{3}{V}_1$-discrete. Let

$$S = U(E_n; n \geqq 1) \subset E_0 ,$$

and let $f\colon S \rightarrow X_1$ be such that, for all x in E_n, $(x, f(x)) \in V_n$ (for each $n \geqq 1$). Thus if $x \in E_m$, $y \in E_n$ are distinct (whether or not $m = n$) we have $(f(x), f(y)) \notin V_1$, for otherwise we should have $(x, y) \in V_m \circ V_1 \circ V_n \subset \overset{3}{V}_1$. Thus f is one-one and $S_1 = f(S)$ is V_1-discrete. By (b) and Theorem 1A, S_1 is also $\mathfrak{U}$-discrete, say $\overset{3}{U}$-discrete where $U \in \mathfrak{U}$ is symmetric. By (i) above there exists N with $V_N \subset U$. Repeating the argument just used, we see that if m, n are both $\geqq N$ then (for $x \neq y$) $x \in E_m$, $y \in E_n$ imply $(x, y) \notin U$, since $V_m \circ U \circ V_n \subset \overset{3}{U}$.

(iv) A finite or countable union of disjoint $\mathfrak{U}$-discrete subsets of E_0 is $\mathfrak{U}$-discrete.

By (iii) it is clearly sufficient to consider the union of two such sets D_1, D_2. As disjoint subsets of E_0, D_1 and D_2 must be $\mathfrak{V}$-remote, hence by (ii) $\mathfrak{U}$-remote; it follows at once that if each is $\mathfrak{U}$-discrete so is their union.

(v) There exists a subset E_{00} of E_0, not itself $\mathfrak{U}$-discrete, such that one at least of any two disjoint subsets of E_{00} is $\mathfrak{U}$-discrete.

It is sufficient to consider the case of subsets which are complementary in E_{00} (and so by (iv) cannot both be $\mathfrak{U}$-discrete). We suppose the proposition false and obtain a contradiction. By induction, there exists (if the proposition is false) a sequence of disjoint subsets of E_0, say $\{E_n, n = 1, 2, \cdots\}$ such that, for each n, neither E_n nor $E_0\backslash(E_1 \cup \cdots \cup E_n)$ is $\mathfrak{U}$-discrete. (If this holds for $n = p$, since $E = E_0\backslash(E_1 \cup \cdots \cup E_p)$ is not of the required type, there exists $E_{p+1} \subset E$ such that neither E_{p+1} nor $E\backslash E_{p+1}$ is $\mathfrak{U}$-discrete.) But this contradicts (iii), which implies that E_n is $\mathfrak{U}$-discrete for all sufficiently large n.

Finally, we write, for all $E \subset X$, $\varphi(E) = 0$ if and only if $E \cap E_{00}$ is $\mathfrak{U}$-discrete, $\varphi(E) = 1$ otherwise. Propositions (iv) and (v) assure us that φ is a countably additive two-valued measure for X, nontrivial since $\varphi(X) = 1$ and $\varphi(F) = 0$ for every finite set F. That is, the cardinal of X must be measurable.

Before applying this theorem to obtain an improved form of Theorem 4A, we prove the following converse.

THEOREM 2. *If $\mathfrak{K}$ is any measurable cardinal, there exists a space $(X, \mathfrak{V})$, X of cardinal $\mathfrak{K}$ and $\mathfrak{V}$ metrisable, and a uniformity $\mathfrak{U}(\neq \mathfrak{V})$ for X such that $\mathfrak{U}, \mathfrak{V}$ are proximity-equivalent but not H-equivalent on X, while their restrictions to $X_1 \times X_1$, where X_1 is a certain dense subset of X, are H-equivalent on X_1.*

Let Y be a set of cardinal $\mathfrak{K}$, A the set of ordinals α, $1 \leqq \alpha \leqq \omega$, and $X = Y \times A$, also of cardinal $\mathfrak{K}$. We define a metric ρ for X, and the associated uniformity $\mathfrak{V}$, by writing

$$\begin{aligned} \rho[(y, \alpha), (y', \alpha')] = {} & 1 \text{ if } y \neq y'; \\ & m^{-1} \text{ if } y = y',\ \alpha = m,\ \alpha' = \omega\ , \\ & \text{or if } y = y', \alpha = \omega, \alpha' = m; \\ & |m^{-1} - n^{-1}| \text{ if } y = y', \alpha = m, \alpha' = n; \\ & 0 \text{ if } y = y', \alpha = \alpha'. \end{aligned}$$

It is clear that this is a metric, and that $\mathscr{T}(\mathfrak{V})$ is the product of the discrete topology on Y and the order topology on A. Let φ be a nontrivial measure for Y with values 0 and 1; write $\mathscr{F} = \{E; E \subset Y$ and $\varphi(E) = 1\}$. We remark that $\mathscr{F}$ is a countably intersective nontrivial ultrafilter over Y. For $E \in \mathscr{F}$ and $1 \leqq n < \omega$ we define (E, n)

as the set of points $[(y, \alpha), (y', \alpha')]$ in $X \times X$ such that *either* $y = y'$ and $\alpha = \alpha'$ *or* $y = y'$ and α, α' both $\geqq n$ *or* again y, y' both in E and α, α' both $\geqq n$. It is easily checked that the system $\{(E, n); E \in \mathscr{F}, 1 \leqq n < \omega\}$ is finitely intersective and is the base of a uniformity for X, which we take for $\mathfrak{U}$. Finally, we put $X_1 = Y \times (A \backslash \{\omega\})$, ρ-dense in X.

The set $\{(y, \omega); y \in Y\}$ is $\mathfrak{V}$-discrete but not $\mathfrak{U}$-discrete; by Theorem 1A $\mathfrak{U}$ and $\mathfrak{V}$ are not H-equivalent on X. We prove that the remaining conditions are satisfied.

(i) $\mathfrak{U}, \mathfrak{V}$ are proximity-equivalent on X.

If P, Q are subsets of X such that $\rho(P, Q) \geqq N^{-1}$, then for each $y \in Y$ the set $\{(y, \alpha); \alpha > N\}$ meets at most one of the sets P, Q. Write $P_0 \subset Y = \{y; \exists \alpha, \alpha > N, (y, \alpha) \in P\}$ and define Q_0 similarly. Since $P_0 \cap Q_0 = \emptyset$, at most one of P_0, Q_0, and hence at least one of $Y \backslash P_0$, $Y \backslash Q_0$, is in $\mathscr{F}$: say $Y \backslash P_0 \in \mathscr{F}$. Then for $(y, \alpha) \in P$ and $(y', \alpha') \in Q$, $[(y, \alpha), (y', \alpha')] \notin (Y \backslash P_0, N + 1)$. Thus P, Q are $\mathfrak{U}$-remote so that $\mathfrak{U}$ is proximity-finer than $\mathfrak{V}$: the reverse relation is trivial. (As $\mathfrak{V}$ is metric we now know that $\mathfrak{U} \subset \mathfrak{V}$, a fact which is easily checked directly.)

(ii) The restrictions of $\mathfrak{U}, \mathfrak{V}$ are H-equivalent on X_1.

Let $P \subset X_1$ be $\mathfrak{V}$-discrete; say $\rho(p, p') \geqq N^{-1}$ if $p \neq p'$ and both are in P. Then for each $y \in Y$ there is at most one m with $m \geqq N$, $(y, m) \in P$. The sets $Y_m = \{y; (y, m) \in P\}$, $N \leqq m < \omega$, are disjoint, so there is at most one such m, say $m = M$, with $Y_m \in \mathscr{F}$. If M exists it is easily checked that P is $(Y_M, M + 1)$-discrete. If no M exists then, since $\varphi(Y_m) = 0$, all $m \geqq N$, $Y_0 = Y \backslash U(Y_m; m \geqq N)$ must be in $\mathscr{F}$; again we check that P is (Y_0, N)-discrete. Thus every $\mathfrak{V}$-discrete subset of X_1 is also $\mathfrak{U}$-discrete; by Theorem 1A, since (i) holds and $\mathfrak{U} \subset \mathfrak{V}$, the restrictions of $\mathfrak{U}, \mathfrak{V}$ are H-equivalent on X_1.

To obtain as wide a generalization as possible of Theorem 4A, we remark that in the statement and proof of Theorem 2A it is essentially irrelevant that $K \subset X$; K may be any compact uniform space (with uniformity $\mathfrak{W}$), in particular, any compact T_2 space with its unique natural uniformity. With a view to a later application, we point out further that when we say that an indexed set $\{y_i; i \in I\}$ is V-discrete, we mean that $(y_i, y_j) \in V$ and $i, j \in I$ imply $y_i = y_j$, not necessarily $i = j$.

THEOREM 3. *Let $(X, \mathfrak{V})$ be a uniform space, $\mathfrak{V}$ having an enumerable base, $(B, \mathfrak{W})$ any precompact uniform space. Suppose there exists a set of functions $\{f_i: B \rightarrow X; i \in I\}$ such that*

(i) $\bigcup(f_i(B); i \in I) = X$;

(ii) *for each* $b \in B$, *the set* $E = \{f_i(b); i \in I\}$ *is* V*-discrete, for some fixed* $V \in \mathfrak{V}$;

(iii) *the functions* f_i, $i \in I$ *form an equi-uniformly continuous set. Then, if (and in general only if) the cardinal of* X *is nonmeasurable,* $\mathfrak{V}$ *is* H*-singular over* X.

COROLLARY. *The theorem holds whenever* I *has nonmeasurable cardinal.*

We omit the details of the proof, which proceeds by extending the functions f_i to map the compact completion of $(B, \mathfrak{W})$ into the completion of $(X, \mathfrak{V})$, almost precisely as in the first part of the proof of Theorem 4A, and then applying Theorem 1. (It is known that if the cardinal $\mathfrak{K}$ of X is nonmeasurable then so is the cardinal of its completion; in this case as $\mathfrak{V}$ is metrisable the completion has cardinal at most $2^{\mathfrak{K}}$.)

To prove the Corollary we observe that, whatever may be the cardinal of B, each $f_i(B)$ is precompact in a metrisable uniformity, hence of cardinal $\mathfrak{C}$, so that by (i) and the properties of cardinals we know that the cardinal of X is nonmeasurable.

If the cardinality condition is dropped, the subspace $(X_1, \mathfrak{V} | (X_1 \times X_1))$ of Theorem 2 provides a counter-example. We take for B the subspace $\{n^{-1}; n = 1, 2, \cdots\}$ of R^1, with the obvious mappings $f_y(n^{-1}) = (y, n) \in X_1$, for each $y \in Y$.

2. A simple sufficient condition for a metric uniformity to be H-singular. The criterion of Theorem 2A is intrinsic for the space concerned, but rather complex. Our remark above, that K need not be a subspace of X, strengthens the theorem but removes its intrinsic character. We can however deduce, in the case when $\mathfrak{V}$ is metrisable, a simple intrinsic criterion sufficient for H-singularity. The idea used, and the basic lemma needed, can be stated without the assumption of metrisability; the rest of the proof is essentially similar to that of the well-known theorem stating that every compact metric space is a continuous image of the Cantor set, though there are minor technical complications.

We say that a uniform space $(X, \mathfrak{V})$ is *equi-uniformly locally totally bounded* (abbreviated as e.l.t.b.)[2]; and in particular V_0-e.l.t.b., if and only if there exists $V_0 \in \mathfrak{V}$ such that, for every $V_1 \in \mathfrak{V}$, the number of (distinct) points in an arbitrary V_0-small and V_1-discrete

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[2] I am indebted to the referee for pointing out that this is equivalent to saying that $\mathfrak{V}$ has a basis defined (in the usual manner) by a star-bounded [**2**, p. 94] collection of coverings of X.

subset of X is bounded. We denote by $N(V_1)$ the greatest such number (for a given V_0). We define similarly a (V_0)-e.l.t.b. subset of X.

LEMMA. *If X is V_0-e.l.t.b., and if V, $V_1 \in \mathfrak{B}$ are symmetric and $\overset{2}{V} \subset V_0$, then there exists a set of at most $N(V_1)$ sets E_n, each V-discrete, such that $\bigcup_n V_1(E_n) = X$.*

Proof. Let E be a maximal V_1-discrete subset of X; since E is maximal $V_1(E) = X$. Let E_1 be a maximal V-discrete subset of E, E_2 of $E \backslash E_1$, E_3 of $E \backslash (E_1 \cup E_2)$ and so on; if and as soon as $E_1 \cup \cdots \cup E_n = E$ we terminate the process. If x is any point of E, $V(x)$ (being V_0-small) contains at most $N(V_1)$ points of E. If, for any m, E_m is defined and $x \notin E_1 \cup \cdots \cup E_m$, then by the maximality condition each of $E_1, \cdots, E_m$ must meet $V(x) \cap E$. Thus $m \leqq N(V_1)$-1, as $x \in E$; hence $x \in E_1 \cup \cdots \cup E_m$ for some $m \leqq N(V_1)$. Since x is arbitrary in E we have, for some $m \leqq N(V_1)$, $E = E_1 \cup \cdots \cup E_m$ and so $X = V_1(E) = V_1(E_1) \cup \cdots \cup V_1(E_m)$

THEOREM 4. *If $(X, \mathfrak{B})$ is a complete* e.l.t.b. *space, and $\mathfrak{B}$ has a countable base, then $\mathfrak{B}$ is H-singular over X.*

COROLLARY. *The same is true if X is not complete, if its cardinal is nonmeasurable.*

We suppose, for convenience, $\mathfrak{B}$ defined by a metric ρ; we write as usual V_ε for $\{(x, y); \rho(x, y) < \varepsilon\}$, $S(E, \varepsilon)$ for $V_\varepsilon(E)$, and say ε-discrete, ε-e.l.t.b. for V_ε-discrete, V_ε-e.l.t.b. Let then X be ε_0-e.l.t.b., and let $\varepsilon_1 = \varepsilon_0/10$. By the lemma, we can find a finite number N_0 of disjoint $5\varepsilon_2$-discrete sets, say E_n, $1 \leqq n \leqq N_0$, such that $\bigcup S(E_n, \varepsilon_1) = X$. We now take a sufficiently large index set I, the same for all n, and index the points of each E_n as $x_i(n)$ (repetitions being allowed but the whole of E_n being covered).

For each integer $p \geqq 1$, let N_p be the maximum number of points in any $2^{-p}\varepsilon_1$-discrete set of diameter at most $2^{2-p}\varepsilon_1 (< \varepsilon_0)$. We define, in succession, for each $x \in E_0 = \bigcup E_n$ and each finite set of indices $n_1, \cdots, n_p$ such that $1 \leqq n_r \leqq N_r$ all r, a point $y(x; n_1 \cdots, n_p)$ in such a way that

(i) $\bigcup (S[y(x; n_1), (1/2)\varepsilon_1]; 1 \leqq n_1 \leqq N_1) \supset S(x, \varepsilon_1)$;

(i)′ $\bigcup (S[y(x; n_1, \cdots, n_p), 2^{-p}\varepsilon_1]; 1 \leqq n_p \leqq N_p)$
$\supset S[y(x; n_1, \cdots, n_{p-1}), 2^{1-p}\varepsilon_1]$ for $p > 1$;

(ii) $\rho[y(x; n_1), x] < \varepsilon_1$;

(ii)′ $\rho[y(x; n_1, \cdots, n_p), y(x; n_1, \cdots, n_{p-1})] < 2^{1-p}\varepsilon_1, p > 1$.

By the definition of N_p this is obviously possible (repetitions being allowed).

Let K (compact) be the product of discrete spaces $D_0, D_1, \cdots, D_p, \cdots$; D_p having N_p members for each $p \geqq 0$. A point k of K may be represented by a sequence of integers $\{k(p); 1 \leqq k(p) \leqq N_p, p = 0, 1, \cdots\}$; the product—topology is induced by the metric $d(k, k') = 2^{-p}$ if and only if p is the least r such that $k(r) \neq k'(r)$ (and of course $d(k, k) = 0$). We define $f_i(k)$ as $\lim_{p\to\infty} y(x_i[k(0)]; k(1), \cdots k(p))$. It follows from our requirements above, by standard arguments, that $f_i(k)$ is defined for all $k \in K$ and that the functions f_i are equi-uniformly continuous from (K, d) into (X, ρ). Moreover, the set $\{f_i(k); k(0) = m\}$ is compact and contained in the closure of $S(x_i(m), 2\varepsilon_1)$ and, being dense (at least) in $S(x_i(m), \varepsilon_1)$, it contains $S(x_i(m), \varepsilon_1)$. We note that, since the points y are defined as functions of the points x, not directly in terms of the indices $i \in I$, if for any $i, j \in I$ we have $x_i(m) = x_j(m)$ then $f_i(k) = f_j(k)$ whenever $k(0) = m$. If however $x_i[k(0)]$ and $x_j[k(0)]$ are distinct then (since E_n is $5\varepsilon_1$ discrete for each n) $\rho(f_i(k), f_j(k)) \geqq 5\varepsilon_1 - 4\varepsilon_1 = \varepsilon_1$. Thus all the conditions of Theorem 1A, as modified by the remarks following Theorem 2, are satisfied, and our theorem is proved.

The corollary follows at once, with the help of Theorem 1, by applying the theorem to the (metric) completion of X, which is clearly also e.l.t.b.

3. Griteria similar to that of Theorem 4.

There seems to be a natural connection, at least for metrisable uniformities, between local total boundedness and H-singularity. The construction of the counter-example in [3] depended essentially on the fact that the space considered was, so to speak, "uniformly locally nontotally-bounded"; one can make this notion precise and show that such a (metric) uniformity is certainly not H-singular. The wide gap between these two opposing criteria may be somewhat narrowed; we give below two theorems which say, very roughly, that in each case a finite number of small portions of the space may be disregarded (as will be seen, the exact expression is rather complicated). I have not however been able to obtain any necessary and sufficient condition for H-singularity. (For simplicity, our results are stated in terms of a given metric.)

THEOREM 5. *If* (X, ρ) *is a complete metric space such that, for each* $\delta > 0$, *there exists a finite set* $E(\delta)$ *with* $X \backslash S(E(\delta), \delta)$ e.l.t.b., *then the uniformity* $\mathfrak{V}$ *defined by* ρ- *is H-singular. The same holds for X not complete, if its cardinal is nonmeasurable.*

Proof. Suppose X ρ-complete, and $\mathfrak{U}$ H-equivalent to $\mathfrak{V}$ on X. Given $\varepsilon > 0$, put $\delta = (1/3)\varepsilon$ and form $E(\delta)$. For each x_m in $E(\delta)$ the sets $S(x_m, \delta)$ and $X \backslash S(x_m, 2\delta)$ are ρ-remote, hence (Theorem 1A) $\mathfrak{U}$-

remote; that is, $\exists U_m \in \mathfrak{U}$ such that if $\rho(x_m, x) < \delta$ and $(x, y) \in U_m$ then $\rho(x_m, y) < 2\delta$ and hence $\rho(x, y) < 3\delta = \varepsilon$. By Theorem 4, $\mathfrak{U}$ and $\mathfrak{V}$ induce identical uniformities over the closed, hence complete, set $X \backslash S(E(\delta), \delta)$. Since $E(\delta)$ is finite it easily follows that for some $U_0 \in \mathfrak{U}$ we have $(x, y) \in U_0 \Rightarrow \rho(x, y) < \varepsilon$, all $x, y \in X$; that is, $\mathfrak{U} \supset \mathfrak{V}$. The reverse inclusion certainly holds since $\mathfrak{V}$ is metric and $\mathfrak{U}, \mathfrak{V}$ are proximity-equivalent.

As before, we deduce the corollary by means of Theorem 1. We remark that it is easy to show by examples that Theorem 5 is effectively stronger than Theorem 4.

Finally, we give a theorem in the opposite direction. Since the construction and proof are very similar to those used in the special case described in (2), they are given in a slightly condensed form.

THEOREM 6. *Let (X, ρ) be a metric space such that, for some $\delta_0 > 0$, there exists in X a $2\delta_0$-discrete sequence $\{x_n; n = 1, 2, \cdots\}$ of distinct points, with the following property; for any $\delta, 0 < \delta \leqq \delta_0$, there exists $\eta = \eta(\delta), 0 < \eta \leqq \delta$, such that, for every integer m and every sequence $\{y_n; n = 1, 2, \cdots\}$ satisfying $S(y_n, \delta) \subset S(x_n, \delta_0)$ for all n, all but a finite number of the sets $S(y_n, \delta)$ contain η-discrete sets A_n each having more than m members. Then the uniformity $\mathfrak{V}$ defined by ρ is not H-singular over X.*

Proof. Define δ_p inductively by $\delta_{p+1} = (1/4)\eta\{(1/2)\delta_p\}$, all $p \geqq 0$ (so that $\delta_p \leqq 2^{-3p}\delta_0$ since $\eta(\delta) \leqq \delta$). If and only if E is a $2\delta_p$-discrete set we define $h(p, E, x)$ as $\max[0, 1 - \delta^{-1}_{p+1}\rho(x, E)]$, and $d_{p,E}(x, y) = h(p, E, x) + h(p, E, y)$, except when there is a point z of E such that x and y are both in $S(z, \delta_{p+1})$, in which case $d_{p,E}(x, y) = |h(p, E, x) - h(p, E, y)|$. We define a uniformity $\mathfrak{U}$ with a sub-base consisting of all sets of one of the forms

(a) $\{(x, y); d_{p,E}(x, y) < \varepsilon\}$, where $\varepsilon > 0$ and E is $2\delta_p$-discrete;

(b) $\{(x, y); |f(x) - f(y)| < \varepsilon\}$, where f is any uniformly continuous function from (X, ρ) to the unit interval $[0, 1]$.

It is easily checked that $\mathfrak{U} \subset \mathfrak{V}$, that $\mathfrak{U}, \mathfrak{V}$ are proximity equivalent (because of the presence of the sets of type (b)), and that any $\mathfrak{V}$-(i.e., ρ-)discrete set E is also $\mathfrak{U}$-discrete, since, for some p, E is $2\delta_p$-discrete. Thus, by Theorem 1A, $\mathfrak{U}$ and $\mathfrak{V}$ are H-equivalent.

It remains to prove that $\mathfrak{U} \neq \mathfrak{V}$. It is sufficient to show that, given any finite set of h-functions, there exists an infinite $(1/2)\delta_0$-discrete set, at all points of which all the h-functions vanish; for as the f-functions are bounded we can apply to them a "pigeon-hole" argument and thus show that, for any given $U \in \mathfrak{U}$, $(x, y) \in U$ cannot imply $\rho(x, y) < (1/2)\delta_0$.

Suppose then that m_0 of the given h-functions have $p = 0$, m_1 have $p = 1$, and so on up to m_q with $p = q$ say. Apply the condition of the enunciation, first with $\delta = (1/2)\delta_0$ and $m = 1 + m_0$, putting $y_n = x_n$. It can be seen, by calculating distances, that for any $2\delta_0$-discrete set E and any given n there is at most one set $S(y, \delta_1)$, $y \in A_n$, which meets $\{x; h(0, E, x) \neq 0\}$. If therefore $n \geqq N_0$ (say) we can choose $x_{n,1} \in A_n$ such that all the m_0 h-functions with $p = 0$ vanish throughout $S(x_{n.1}, \delta_1)$: moreover $S(x_{n,1}, \delta_1) \subset S\{x_n, (3/4)\delta_0\}$. We repeat the argument with $y_n = x_{n,1}$ for $n \geqq N_0$ (and, say, $y_n = x_n$ for $n < N_0$), putting $m = m_1 + 1$, $\delta = (1/2)\delta_1$, and so on. Finally we obtain a set of points $\{x_{n,q+1}; n \geqq N_q\}$ at which all the given h-functions vanish; since $x_{n,q+1} \in S\{x_n, (3/4)\delta_0\}$ the set $\{x_{n,q+1}\}$ is $(1/2)\delta_0$-discrete.

As an example of the application of Theorem 6, let X_0 be (cf. [3]) the set of all bounded real sequences $x = (x_0, x_1, x_2, \cdots)$ with the metric $\rho(x, x') = \sup | x_n - x'_n |$, and let X_r be the subset of X_0 defined by $x_0 = r$, $0 \leqq x_n \leqq 1$ for $1 \leqq n \leqq r$, $X_n = 0$ for $n > r$. The subspace $X = \bigcup (X_r; r = 1, 2, \cdots)$ satisfies the conditions of Theorem 6, so that the uniformity defined by ρ is not H-singular over X. We note that X is locally compact and σ-compact, so that a metric uniformity may have quite a 'good' topology and yet not be H-singular.

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Received June 30, 1966. This paper was written while the author held a visiting position at the University of Washington.

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PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics
Vol. 28, No. 1 March, 1969