ON SUBGROUPS OF FIXED INDEX

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If \( k \in \mathcal{H} \), where \( \mathcal{H} \) is a subgroup of a group \( \mathcal{I} \), then closure implies \( k, k^2, \ldots \in \mathcal{H} \). Nonempty subsets \( S \subset \mathcal{I} \) with the inverse property \( s^m \in S \) implies \( s, s^2, \ldots, s^m \in S \) \((m = 1, 2, \ldots)\) will be called stellar sets. Let \( p^a \) be a fixed prime power. If a stellar set \( S \) of an abelian group \( \mathcal{I} \) intersects every subgroup \( \mathcal{K} \) of index \( p^a \) in \( \mathcal{I} \), and \( 0 \notin S \), then the cardinal \( |S| \) of \( S \) is bounded below by \( p^a \) (Theorem 3), when \( \mathcal{I} \) satisfies a mild condition.

Hence for instance a subset \( S \) of euclidean \( n \)-space \( E_n \) intersecting all sublattices of determinant \( p^a \) of the fundamental lattice will have at least \( p^a \) elements, and more if no element is divisible by \( p^a \).

Henceforth \( \mathcal{I} \) will always be an additive abelian group, so a stellar set will be one with

\[
\emptyset \neq S \subset \mathcal{I} \\
mg \in S \Rightarrow g, 2g, \ldots, mg \in S \ (g \in \mathcal{I}, \ m = 1, 2, \ldots).
\]

Examples of stellar sets are \( \mathcal{I} \) itself, and its periodic part \([5, \text{p. 137}]\); and a star set \([7]\) is a symmetric stellar set. There are stellar sets of one element \( s \), i.e., those \( s \) for which \( s = mg(m = 1, 2, \ldots) \) implies \( m = 1 \). Now let \( p \) be a fixed prime, and suppose \( S \) intersects every subgroup \( \mathcal{K} \) of \( \mathcal{I} \) of index \( p \). Suppose also

\[
0 \notin S
\]

(if \( 0 \notin S \) the intersection property is redundant). Then we can say the following (in this paper we denote \( |A| = \text{cardinal of} \ A \), \( mA = \{ma; \ a \in A\} \), for any set \( A \) and integer \( m \)):

**Theorem 1.** Let \( p \) be a fixed prime, \( \mathcal{I} \) an abelian group, and \( S \) a stellar set with \( 0 \notin S \) which intersects all subgroups \( \mathcal{K} \) of index \( \mathcal{I}: \mathcal{K} = p \). Then

\[
|S| \geq p.
\]

When \( S \cap p\mathcal{I} = \emptyset \) we have \( |S| > p \).

A similar result holds for ordinary sets \( T \):

**Theorem 2.** Suppose \( p \) is a fixed prime, \( \mathcal{I} \) is an abelian group with more than one subgroup of index \( p \), and \( T \) is any subset of \( \mathcal{I} \) with
(4) \[ T \cap pS = \emptyset \]

which intersects all subgroups \( S \) of index \( S : S = p \). Then

(5) \[ |T| \geq p + 1. \]

When \( S \) is the fundamental lattice \( \Lambda_0 \) [2, 4] in \( r \)-space \( E_r \) of all points with integral coordinates, Theorems 1 and 2 are immediate using Rogers’ proof of his Theorem 1 [7] on starsets, the small adjustment needed being clear. He states his theorem with a slightly stronger hypothesis equivalent to “\( S \) intersects all subgroups of index \( \leq p \)”, and for this more stringent requirement Cassels [3], replacing \( p \) by \( n \), has made elegant use of a generalization of Bertrand’s postulate due to Sylvester [9] and Schur [8] to show \( |S| \geq n \) for \( n = 1, 2, \ldots \) and any stellar set \( S \) of an abelian \( S \) with no periodic part. For \( n = p^a \) a prime power we shall extend this as follows:

**Theorem 3.** Suppose that \( n = p^a \) is fixed, \( S \) is an abelian group containing no element of order \( p^b \) when \( 1 < p^b < p^a \), and that \( S \) is a stellar set with \( 0 \in S \) which intersects all subgroups \( S \) of index \( S : S = p^a \). Then

(6) \[ |S| \geq p^a. \]

When \( S \cap p^aS = \emptyset \), we have

\[ |S| \geq p^a + \left\{ \begin{array}{ll} p & \text{if } \alpha > 1, \\ 1 & \text{if } \alpha = 1. \end{array} \right. \]

Note the requirement “contains at least one subgroup of index \( p^a \)” is a natural one, but it is an unneeded restriction on \( S \). Note also that Theorem 1 is an immediate consequence of Theorem 3.

2. A lemma. We find it useful, for Rogers’ case \( S = \Lambda_0 \subset E_r \), to restate Theorem 3 in altered form. We denote \( \bar{x} = (x_1, \ldots, x_r) \) so that

\[ \Lambda_0 = \{ \bar{x} : \text{all the } x_i \text{ are integers, } i = 1, \ldots, r \}, \]

and \( S = \Lambda_0 \) is isomorphic to a direct sum of \( r \) infinite cyclic groups. When \( \bar{x} \in \Lambda_0 \) we define \( p\mid \bar{x} \) to mean \( p\mid x_i, \ldots, p\mid x_r \), and

\[ ||x||_p = \max \{ \alpha : p^\alpha \mid \bar{x} \}. \]

Let \( T \) be any subset of \( \Lambda_0 \) satisfying

(7) \[ p^a \Lambda_0 \cap T = \emptyset \quad (T \subset \Lambda_0), \]

and a modified stellar condition
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\[
\begin{align*}
\{p^βx \in T & \implies x, 2x, \cdots, p^βx \in T \\
(1 \leq β \leq α, p^α \text{ fixed})
\end{align*}
\]

and consider congruences

\[
(9) \quad \bar{l} \cdot x = l_1x_1 + \cdots + l_rx_r \equiv 0(p^α)(\bar{l} \in \Lambda, p \not| \bar{l})
\]

**Lemma.** If \( T \subset \Lambda \) satisfies (7) and (8), \( r \geq 2 \) and the congruence (9) has for each \( \bar{l} \) a solution \( x \in T \), then \( T \) contains at least \( p^α + p^{\min(α,β)-1} \) distinct elements mod \( p^α \),

\[
| T \mod p^α | \geq p^α + \begin{cases} p & \text{if } α > 1 \\
1 & \text{if } α = 1 \end{cases}
\]

**Proof.** We consider two cases, (i) \( α = 1 \) or \( r \leq α \), and (ii) \( r > α \geq 2 \).

For the first case, a simple counting argument will suffice. Define

\[
θ(i, α) = \frac{p^{(i-1)(α-1)}(p^i - 1)}{p - 1}
\]

Then there are exactly

\[
\sum_{k=1}^{r} p^{(α-1)(k-1)+α(r-k)} = θ(r, α)
\]

distinct congruences (9), representable by

\[
\bar{l} = (pm_1, \cdots, pm_{k-1}, 1, l_{k+1}, \cdots, l_r)
\]

If \( \bar{y} \equiv b\bar{x} \mod p^α \) then clearly \( \bar{y} \) satisfies every congruence \( \bar{x} \) does, and hence we may construct a subset \( V \) of \( T \) which likewise satisfies every congruence (9), and also

\[
\begin{align*}
\{ & \bar{x} \in V, \bar{y} \in V, \bar{y} \equiv b\bar{x} \mod p^α \Rightarrow \bar{y} = \bar{x} , \\
& \bar{x} \in V \Rightarrow \bar{x} \text{ satisfies some congruence (9)} \}
\end{align*}
\]

Any \( \bar{x} \in V \) may be expressed as

\[
\bar{x} = \bar{x}'p^i(p \not| \bar{x}'; 0 ≤ i = \| \bar{x} \|_p < α)
\]

by (7), since \( V \subset T \). A fixed \( \bar{x} \in V \) obeys (9) for at least one \( \bar{l} \) and in fact for precisely those \( \bar{l} \) satisfying \( \bar{l} \cdot \bar{x}' \equiv 0(p^{α-i}) \); these correspond to exactly \( p^iθ(r - 1, α) \) congruences (consider, e.g., \( \bar{x}' = (1, 0, \cdots, 0) \)). Hence, counting over the \( θ(r, α) \) congruences (9), we get

\[
θ(r, α) \leq \sum_{\bar{x} \in V} p^{|\bar{x}|}rθ(r - 1, α)
\]

Now \( \bar{x} \in V \) obeys (8), since \( V \subset T \). Hence to each \( \bar{x} = \bar{x}'p^i \) in \( V \) there correspond \( p^i \) elements
(14) \[ \mathcal{I}(\bar{x}) = \{ \lambda \bar{x}' : \lambda = 1, \ldots, p^j \} \subset \mathcal{T} \quad (\bar{x} \in \mathcal{V}) . \]

Moreover,

(15) \[ \bar{x}_1 \neq \bar{x}_2 \text{ implies } \mathcal{T}(\bar{x}_1) \cap \mathcal{T}(\bar{x}_2) = \emptyset \quad (\bar{x}_1, \bar{x}_2 \in \mathcal{V}) , \]

for otherwise \( \lambda_i \bar{x}_1 = \lambda_i \bar{x}_2 \), \( \lambda_i = \lambda_i' p^{\gamma_i} (p \nmid \lambda_i') \), without loss of generality \( \theta = \theta_1 - \xi, (\theta_1 - \xi) \geq 0 \), and \( \lambda_i' \bar{x}_1 = \lambda_i' p^{\gamma_i} \bar{x}_i \), \( \bar{x}_2 = (\lambda_i')^{-1} \lambda_i' p^{\gamma_i} \bar{x}_i \bmod p^\gamma \), \( \bar{x}_2 = \bar{x}_1 \) by (12). Thus by (13), (15),

\[ |T| \geq \sum_{\bar{x} \in \mathcal{V}} p^{\gamma |x|} \geq \theta(r, \alpha)/\theta(r - 1, \alpha) \]

\[ = p^r + \frac{p^{a - 1}(p - 1)}{p^{r - 1} - 1} . \quad (r \geq 2) \]

If \( \alpha = 1 \) we have \( |T| \geq p + (p - 1)(p^{r - 1} - 1)^{-1} > p \), so \( |T| \geq p + 1 \); if \( r \leq \alpha > 1 \) then

\[ |T| - p^r \geq p^{a - 1}(p - 1)(p^{a - 1} - 1)^{-1} > p - 1 , \]

\( |T| - p^r \geq p \), and case (i) is verified.

For our second case \( r > \alpha \geq 2 \) we employ induction on \( r \). Let \( r = j \), define \( \mathcal{V} \subset \mathcal{T} \) as in case (i), and denote

(16) \[ \bar{x} = (x_1, \ldots, x_{j-1}, x_j) = (\bar{x}_o, x_j) . \]

There are \( p^{r-1} + \cdots + p + 1 \geq p^a + p + 1 \) subgroups

\[ H(H') = \{ \lambda \bar{a}' \bmod p : \lambda = 1, \ldots, p \equiv 0 \} \]

(\( a' \) fixed, \( p \nmid a' \)), any two of which intersect in a point \( \bar{x} \) divisible by \( p \). So if \( V \) contains a primitive \( (p \nmid \bar{x}) \) point from each subgroup, we have \( |V| \geq p^a + p + 1 \) and our result follows. Hence we may assume that \( V \) does not intersect some \( H(a') \), where without loss of generality \( a' = (0, \ldots, 0, 1) \); then \( V \) contains no point of type \( \bar{x} = \lambda(p \bar{y}, 1) \bmod p \) when \( p \nmid \lambda \), and hence by (8) no such point for any \( \lambda = 1, 2, \ldots \),

(17) \[ \bar{x} \in \mathcal{V} \implies \bar{x} = p^\beta (\bar{y}, y_j) . \quad (p \nmid \bar{y}, 0 \leq \beta < \alpha) . \]

Now define sets \( \mathcal{T}(\bar{x}) \) as in (14) and denote their union by \( W \),

\[ W = \cup \{ \mathcal{T}(\bar{x}) : \bar{x} \in \mathcal{V} \} , \]

so that \( \mathcal{V} \subset W \subset \mathcal{S} \), and \( W \) is the (smallest) set generated by \( V \) which satisfies the modified stellar condition (8). Denote

(18) \[ W_0 = \{ \bar{x}_o : (\bar{x}_o, x_j) \in W \text{ for some } x_j \} . \]

Then by (17), (18), points \( \bar{x}_o p^\beta (p \nmid \bar{x}_o') \) of \( W_0 \) correspond to points \( p^\beta (\bar{x}_o, x_j) \) of \( W \) and so clearly \( W_0 \) satisfies (7) and (8). But \( V \) and
hence \( W \) satisfies every congruence \( \bar{l} \) in (9); thus \( W \) and hence \( W_0 \) satisfies every \( \bar{l} \) with \( l_j = 0 \) for some \( \bar{x}_0 = (x_1, \ldots, x_{j-1}) \in W_0 \) such that

\[
l_1x_1 + \cdots + l_{j-1}x_{j-1} \equiv 0 (p^\alpha) \quad (l_1, \ldots, l_{j-1}, p) = 1.
\]

Thus by our induction hypothesis (\( r = j - 1, \alpha \geq 2 \)) there are at least \( p^\alpha + p \) such \( \bar{x}_0 \in W_0 \), and

\[
|S| \geq |W| \geq |W_0| \geq p^\alpha + p.
\]

As our result is already established for \( r = \alpha \) (case (i)), this completes the proof of the lemma.

3. Proof of Theorems 2 and 3. Consider the homomorphism \( \eta: \mathcal{S} \rightarrow \mathcal{F}/p^\alpha \mathcal{F} \)

(cf. Cassels [3] for his case \( s = 1 \)); for Theorem 2 we take \( \alpha = 1 \).

We see easily that if \( \mathcal{S}: \mathcal{X} = p^\alpha \) then \( p^\alpha \mathcal{S} \subset \mathcal{X} \) and so there is a one-to-one correspondence between all \( \mathcal{X}, \mathcal{X}' \) of index \( p^\alpha \) in \( \mathcal{S}, \mathcal{F} \) respectively; and any subset \( V \) of \( \mathcal{S} \) intersects all such \( \mathcal{X} \) if and only if \( \bar{V} \) intersects all such \( \mathcal{X}' \) (index \( p^\alpha \)). If \( V \) has the stellar set property this may, however, be lost under \( \eta \). Since \( p^\alpha \mathcal{F} = 0 \) we have by a result of Prüfer [1] that \( \mathcal{F} \) is a direct sum of cyclic groups \( C_i \) of orders \( p^{\beta_i} \leq p^\alpha \); in fact, \( \beta_i = \alpha \) since in all our 3 theorems \( \mathcal{S} \) has no element of order \( p^\beta (0 < \beta < \alpha) \) and hence \( p^{\beta_i}c_i = 0 \) implies \( \beta_i \geq \alpha \). Thus

\[
\mathcal{F} = \sum_{i \in I} C_i (C_i \cong \langle e: p^\alpha e = 0 \rangle).
\]

Note that all \( s \in S \) have infinite period,

\[
ms \neq 0 \quad (s \in S, m = \pm 1, \pm 2, \cdots)
\]

since otherwise \( |m|s = 0, s = (|m| + 1)s \in S \) so \( 0 = |m|s \in S \) contrary to (2). Now suppose \( \bar{0} \in \bar{S} \). Then \( p^\beta g \in S \) so \( g, 2g, \cdots, p^\beta g \in S, |S| \geq p^\alpha \) since otherwise \( ig = ig(i < j) \) and \( g \in S \) has finite period. It remains therefore to settle the matter when

\[
\bar{0} \in \bar{S} \quad (i.e., S \cap p^\alpha \mathcal{S} = \emptyset).
\]

The cases \( |I| = 0, 1 \) in (20) correspond to groups \( \mathcal{S} \) with no, exactly one subgroup of index \( p^\alpha \). In the latter event we have \( \bar{0} \in \bar{S} \), a case already settled. If \( |I| = 0 \) in Theorem 3 then \( \mathcal{S} = p^\alpha \mathcal{S} \) and all stellar sets \( S \) vacuously satisfy the intersection condition. No stellar set is empty, so we have \( s \in S, s = p^\alpha s_1, \ s_1 = p^\alpha s_2, \cdots, \) and \( |S| = \infty \) since otherwise \( s_i = s_j \) \((i < j)\) and \( s_j \in S \) has finite period, contrary to (21).
The case $|I| \leq 1$ does not occur for Theorem 2, since here $\mathcal{S}$ has $\geq 2$ subgroups of index $p^\alpha$. Hence we may assume

\[(23) \quad |I| \geq 2.\]

From (23) it is immediate that $\mathcal{S}$ contains more than one subgroup of index $p^\alpha$. We consider only Theorem 3 from now on; Theorem 2 will follow by the same reasoning ($\alpha = 1$).

It remains, then, to verify Theorem 3 when (22), (23) hold. Assume now then

\[(24) \quad |S| < \infty,\]

since if $|S| = \infty$ we have nothing to prove. Then if we decompose $\bar{s} = \sum_{i \in I}$ in (20) we have $s_i \neq 0$ for some $\bar{s} \in \bar{S}$ for only a finite number of $i \in I$, which we may include in a finite set $i = 1, \ldots, j \ (2 \leq j \leq |I|)$. Then

\[
\bar{S} \subset \mathcal{S}^{(0)} \cong A_0 \mod p^\alpha \quad \text{(in j-space $E^j$)}, \quad (2 \leq j),
\]

\[
\mathcal{T} = \mathcal{S}^{(0)} \oplus \mathcal{S}^*,
\]

and we may represent any $\bar{x} \in \mathcal{T}$ uniquely by

\[
\bar{x} = x^{(0)} + x^* = (x_1, \ldots, x_j; x^*) \mod p^\alpha.
\]

The following subgroups $\mathcal{K}$ have index $p^\alpha$ in $\mathcal{T}$ and hence are intersected by $\bar{S}$:

\[
\mathcal{K} = \{ \bar{x} : l_i x_i + \cdots + l_j x_j \equiv 0(p^\alpha) \} \quad (l_1, \ldots, l_j, p) = 1,
\]

where $(l_i, p) = 1$ for some $i$ and $l_i, \ldots, l_j$ are fixed for each $\mathcal{K}$ (cf. [3, preceding (10)]); we have $p \nmid l_i$ for at least one $i$ and so for each $\bar{x} \in \mathcal{K}$, $x_i = -\sum_{j \neq i} l_i^{-1} l_j x_j$. Hence $|\mathcal{K}_0| = p^\alpha(j-1)$,

\[
\mathcal{S} : \mathcal{K} = \mathcal{S}_0 : \mathcal{K}_0 = p^\alpha j/p^{\alpha(j-1)} = p^\alpha.
\]

Elements $\bar{s}$ of $\bar{S}$ are of type $\bar{s} = (s_i, \ldots, s_j; 0^*)$; since $S$ is a stellar set the modified property (8) holds for $T = \bar{S}$; also, $0 = (0, \ldots, 0, 0^*) \in \bar{S}$ and $r = j \geq 2$ by (22), (23). So we may apply the lemma to find there are at least $p^\alpha + p^{\min\{\alpha, 2\} - 1}$ distinct points $(s_i, \ldots, s_j, 0^*)$ in $\bar{S}$; hence

\[
|S| \geq |\bar{S}| \geq p^\alpha + p^{\min\{\alpha, 2\} - 1},
\]

and our proof of Theorems 2, 3 is complete.

4. Remarks. 1. In our proof of Theorem 3 we utilize the stellar property of $S$ only through its consequence in $\bar{S}$, a condition of type (8) with $T = \bar{S}$ which would clearly follow from imposing
condition (8) on $S$, along with $S \neq \emptyset$. Hence we may make the following extension:

**Theorem 4.** Theorem 3 holds for $S$ not a stellar set, if $S$ satisfies (8) ($T = S \subset \mathcal{S}$, $\bar{x} \in \mathcal{S}$), and $S \neq \emptyset$.

2. When $\mathcal{S}$ is not abelian, Theorems 1–4 need not hold; e.g., the direct sum $\mathcal{S} = C^\infty \oplus A_5$ of the infinite cyclic group and alternating group of 60 elements has only one subgroup of index 3, $\mathcal{S} = 3C^\infty \oplus A_5$, and $\mathcal{S}$ is intersected by the stellar set of one element,

$$S = \{3 + \text{cycle } (123)\} \neq 3g.$$  

3. In the excluded case $0 \in S$ the least stellar set containing 0 is the periodic part of $\mathcal{S}$, and $|S| \geq p$ need not follow.

4. When $\mathcal{S} = A_\alpha (r \geq 2)$, the set of all $(1, x_1, 0, \ldots, 0), (px_1, 1, 0, \ldots, 0) \mod p^x$ is a stellar set of $p^x + p^{x-1}$ elements intersecting all congruences (9) mod $p^n$. So our bounds are best possible, for the lemma, when $\alpha = 1, 2$ ($r \geq 2$).

5. In Theorem 3 we must exclude elements of order $p^\beta (\beta < \alpha)$. For consider, e.g., $\mathcal{S} = C^\infty \oplus C\langle \alpha \rangle$ (any $\alpha$). Here the bound is $p^x + 1$.

6. Let $\alpha \geq 2$, $S$ be a stellar set in Euclidean $n$-space $\{\bar{x} = (x_1, \ldots, x_n)\}$ with fewer than $p^x + p$ elements, and no element $p^x \bar{x}$. Then there is a sublattice of the fundamental lattice of determinant $p^x$ (see [2], p. 10) which is not intersected by $S$.

7. Our condition (A)$S$ intersects all subgroups of index $n''$ is equivalent to (B)$\ldots$ index $d; d|n''$ though weaker than (C)$\ldots$ index $m; m < n''$. The latter remark follows from the example $S = \{(4,1), (2,1), (2,0), (1,0)\}$ in $\mathcal{S} = C^\infty \oplus C\langle 2 \rangle$ ($n = 4$). For the former prove first for $d = n/p$ and then iterate: if $\mathcal{S} : \mathcal{H} = n/p$ ($p|n$) and (A) holds then $\mathcal{H} \neq p\mathcal{H}$, there exist $\mathcal{M}$ in $\mathcal{H}$ with $\mathcal{H} : \mathcal{M} = p$ so $\mathcal{S} : \mathcal{H} = n, \mathcal{H} \cap S \neq \emptyset$.

8. Theorem 3 does not hold for all $n = 1, 2, \ldots$. Mr. George M. Bergman of Cambridge, Mass. has kindly furnished me with a set of counterexamples for $\mathcal{S} = C^\infty \oplus C^\infty$, which includes a stellar set $S$ of 76 elements that intersects every subgroup of index 77.

9. Finally, we should like to acknowledge here some parallel though independent work of Mr. Bergman who in unpublished cor-
respondence proves a simpler version of Theorem 4, obtaining a slightly lower bound \((p^a \text{ rather than } p^a + p, 1)\). His proof is in essence similar to ours, except there is no induction step: a homomorphism \(\eta\) (19) reduces the problem to Rogers' case \(\mathcal{E} = \Lambda_0\), and a version of our lemma is proved by arguments resembling ours for \(\alpha = 1\) or \(r \geq \alpha\).

Mr. Bergman in effect considering congruences (9) with \(l_1 = 1\) to obtain his bound \(p^a\) for (10) for all \(r, \alpha\), without induction. We thank Mr. Bergman for the material communicated; among other things it helped remind us to include Theorem 4. We thank him also for welcome suggestions concerning our final draft.

References


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