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If $k \in \mathscr{K}$, where $\mathscr{K}$ is a subgroup of a group $\mathscr{S}$, then closure implies $k^2, k^3, \cdots, \in \mathscr{K}$. Nonempty subsets $S \subset \mathscr{S}$ with the inverse property $s^m \in S$ implies $s, s^2, \cdots, s^m \in S$ ($m = 1, 2, \cdots$) will be called *stellar sets*. Let p^α be a fixed prime power. If a stellar set S of an *abelian* group $\mathscr{S}$ intersects every subgroup $\mathscr{K}$ of index p^α in $\mathscr{S}$, and $0 \notin S$, then the cardinal $|S|$ of S is bounded below by p^α (Theorem 3), when $\mathscr{S}$ satisfies a mild condition.

Hence for instance a subset S of euclidean n-space E_n intersecting all sublattices of determinant p^α of the fundamental lattice will have at least p^α elements, and more if no element is divisible by p^α.

Henceforth $\mathscr{S}$ will always be an additive abelian group, so a *stellar set* will be one with

$$\begin{gathered} \emptyset \neq S \subset \mathscr{S} \\ mg \in S \Rightarrow g, 2g, \cdots, mg \in S (g \in \mathscr{S}, m = 1, 2, \cdots) . \end{gathered} \tag{1}$$

Examples of stellar sets are $\mathscr{S}$ itself, and its *periodic part* [5, p. 137]; and a *star set* [7] is a symmetric stellar set. There are stellar sets of one element s, i.e., those s for which $s = mg (m = 1, 2, \cdots)$ implies $m = 1$. Now let p be a fixed prime, and suppose S intersects every subgroup $\mathscr{K}$ of $\mathscr{S}$ of index p. Suppose also

$$0 \notin S \tag{2}$$

(if $0 \in S$ the intersection property is redundant). Then we can say the following (in this paper we denote $|A| =$ cardinal of A, $mA = \{ma; a \in A\}$, for any set A and integer m):

THEOREM 1. *Let p be a fixed prime, $\mathscr{S}$ an abelian group, and S a stellar set with $0 \notin S$ which intersects all subgroups $\mathscr{K}$ of index $\mathscr{S} : \mathscr{K} = p$. Then*

$$|S| \geqq p . \tag{3}$$

When $S \cap p\mathscr{S} = \emptyset$ we have $|S| > p$.

A similar result holds for ordinary sets T:

THEOREM 2. *Suppose p is a fixed prime, $\mathscr{S}$ is an abelian group with more than one subgroup of index p, and T is any subset of $\mathscr{S}$ with*

$$T \cap p\mathscr{S} = \varnothing \tag{4}$$

which intersects all subgroups $\mathscr{K}$ *of index* $\mathscr{S} : \mathscr{K} = p$. *Then*

$$|T| \geqq p + 1 . \tag{5}$$

When $\mathscr{S}$ is the fundamental lattice Λ_0 [2, 4] in r-space E_r of all points with integral coordinates, Theorems 1 and 2 are immediate using Rogers' proof of his Theorem 1 [7] on starsets, the small adjustment needed being clear. He states his theorem with a slightly stronger hypothesis equivalent to "S intersects all subgroups of index $\leqq p$", and for this more stringent requirement Cassels [3], replacing p by n, has made elegant use of a generalization of Bertrand's postulate due to Sylvester [9] and Schur [8] to show $|S| \geqq n$ for $n = 1, 2, \cdots$ and any stellar set S of an abelian $\mathscr{S}$ with no periodic part. For $n = p^\alpha$ a prime power we shall extend this as follows:

THEOREM 3. *Suppose that* $n = p^\alpha$ *is fixed,* $\mathscr{S}$ *is an abelian group containing no element of order* p^β *when* $1 < p^\beta < p^\alpha$, *and that* S *is a stellar set with* $0 \notin S$ *which intersects all subgroups* $\mathscr{K}$ *of index* $\mathscr{S} : \mathscr{K} = p^\alpha$. *Then*

$$|S| \geqq p^\alpha . \tag{6}$$

When $S \cap p^\alpha \mathscr{S} = \varnothing$, *we have*

$$|S| \geqq p^\alpha + \begin{cases} p & \text{if } \alpha > 1 , \\ 1 & \text{if } \alpha = 1 . \end{cases}$$

Note the requirement "contains at least one subgroup of index p^α" is a natural one, but it is an unneeded restriction on S. Note also that Theorem 1 is an immediate consequence of Theorem 3.

2. A lemma. We find it useful, for Rogers' case $\mathscr{S} = \Lambda_0 \subset E^r$, to restate Theorem 3 in altered form. We denote $\bar{x} = (x_1, \cdots, x_r)$ so that

$$\Lambda_0 = \{\bar{x}: \text{ all the } x_i \text{ are integers}, \quad i = 1, \cdots, r\} ,$$

and $\mathscr{S} = \Lambda_0$ is isomorphic to a direct sum of r infinite cyclic groups. When $\bar{x} \in \Lambda_0$ we define $p|\bar{x}$ to mean $p|x_1, \cdots, p|x_r$, and

$$||x||_p = \max \{\alpha : p^\alpha | \bar{x}\} .$$

Let T be any subset of Λ_0 satisfying

$$p^\alpha \Lambda_0 \cap T = \varnothing \qquad (T \subset \Lambda_0) , \tag{7}$$

and a modified stellar condition

$$(8)\qquad \begin{cases} p^{\beta}\bar{x} \in T \quad \text{implies} \quad \bar{x}, 2\bar{x}, \cdots, p^{\beta}\bar{x} \in T \\ (1 \leqq \beta \leqq \alpha,\ p^{\alpha} \text{ fixed}) , \end{cases}$$

and consider congruences

$$(9)\qquad \bar{l} \cdot \bar{x} = l_1x_1 + \cdots + l_rx_r \equiv 0(p^{\alpha})(\bar{l} \in \Lambda_0,\ p \nmid \bar{l}) .$$

LEMMA. *If $T \subset \Lambda_0$ satisfies* (7) *and* (8), $r \geqq 2$ *and the congruence* (9) *has for each* $\bar{l}$ *a solution* $\bar{x} \in T$, *then* T *contains at least* $p^{\alpha} + p^{\min(\alpha,2)-1}$ *distinct elements* mod p^{α},

$$(10)\qquad |T \bmod p^{\alpha}| \geqq p^{\alpha} + \begin{cases} p & \text{if}\ \alpha > 1 , \\ 1 & \text{if}\ \alpha = 1 . \end{cases}$$

Proof. We consider two cases, (i) $\alpha = 1$ or $r \leqq \alpha$, and (ii) $r > \alpha \geqq 2$. For the first case, a simple counting argument will suffice. Define

$$(11)\qquad \theta(i, \alpha) = \frac{p^{(i-1)(\alpha-1)}(p^i - 1)}{p - 1} .$$

Then there are exactly

$$\sum_{k=1}^{k=r} p^{(\alpha-1)(k-1)+\alpha(r-k)} = \theta(r, \alpha)$$

distinct congruences (9), representable by

$$\bar{l} = (pm_1, \cdots, pm_{k-1}, 1, l_{k+1}, \cdots, l_r) .$$

If $\bar{y} \equiv b\bar{x}$ mod p^{α} then clearly $\bar{y}$ satisfies every congruence $\bar{x}$ does, and hence we may construct a subset V of T which likewise satisfies every congruence (9), and also

$$(12)\qquad \begin{cases} \bar{x} \in V, \bar{y} \in V, \bar{y} \equiv b\bar{x} \bmod p^{\alpha} \Rightarrow \bar{y} = \bar{x} , \\ \bar{x} \in V \Rightarrow \bar{x} \quad \text{satisfies some congruence (9)} . \end{cases}$$

Any $\bar{x} \in V$ may be expressed as

$$\bar{x} = \bar{x}'p^{\xi}(p \nmid \bar{x}';\ 0 \leqq \xi = ||\bar{x}||_p < \alpha)$$

by (7), since $V \subset T$. A fixed $\bar{x} \in V$ obeys (9) for at least one $\bar{l}$ and in fact for precisely those $\bar{l}$ satisfying $\bar{l} \cdot \bar{x}' \equiv 0(p^{\alpha-\xi})$; these correspond to exactly $p^{\xi}\theta(r - 1, \alpha)$ congruences (consider, e.g., $\bar{x}' = (1, 0, \cdots, 0)$). Hence, counting over the $\theta(r, \alpha)$ congruences (9), we get

$$(13)\qquad \theta(r, \alpha) \leqq \sum_{\bar{x} \in V} p^{||\bar{x}||_p}\theta(r - 1, \alpha) .$$

Now $\bar{x} \in V$ obeys (8), since $V \subset T$. Hence to each $\bar{x} = \bar{x}'p^{\xi}$ in V there correspond p^{ξ} elements

$$T(\bar{x}) = \{\lambda\bar{x}' : \lambda = 1, \cdots, p^{\xi}\} \subset T \qquad (\bar{x} \in V) . \tag{14}$$

Moreover,

$$\bar{x}_1 \neq \bar{x}_2 \quad \text{implies} \quad T(\bar{x}_1) \cap T(\bar{x}_2) = \varnothing \qquad (\bar{x}_1, \bar{x}_2 \in V) , \tag{15}$$

for otherwise $\lambda_1\bar{x}_1 = \lambda_2\bar{x}_2$, $\lambda_i = \lambda_i' p^{\theta_i}(p \nmid \lambda_i')$, without loss of generality $\theta = \theta_1 - \xi_1 - (\theta_2 - \xi_2) \geqq 0$, and $\lambda_2'\bar{x}_2 = \lambda_1' p^{\theta}\bar{x}_1$, $\bar{x}_2 \equiv (\lambda_2')^{-1}\lambda_1' p^{\theta}\bar{x}_1 \bmod p^{\alpha}$, $\bar{x}_2 = \bar{x}_1$ by (12). Thus by (13), (15),

$$\begin{aligned} |T| &\geqq \sum_{\bar{x} \in V} p^{||\bar{x}||_p} \geqq \theta(r, \alpha)/\theta(r-1, \alpha) \\ &= p^{\alpha} + \frac{p^{\alpha-1}(p-1)}{p^{r-1}-1} . \qquad (r \geqq 2) \end{aligned}$$

If $\alpha = 1$ we have $|T| \geqq p + (p-1)(p^{r-1}-1)^{-1} > p$, so $|T| \geqq p + 1$; if $r \leqq \alpha > 1$ then

$$|T| - p^{\alpha} \geqq p^{\alpha-1}(p-1)(p^{\alpha-1}-1)^{-1} > p - 1 ,$$

$|T| - p^{\alpha} \geqq p$, and case (i) is verified.

For our second case $r > \alpha \geqq 2$ we employ induction on r. Let $r = j$, define $V \subset T$ as in case (i), and denote

$$\bar{x} = (x_1, \cdots, x_{j-1}, x_j) = (\bar{x}_0, x_j) . \tag{16}$$

There are $p^{j-1} + \cdots + p + 1 \geqq p^{\alpha} + p + 1$ subgroups

$$H(\bar{a}') = \{\lambda\bar{a}' \bmod p : \lambda = 1, \cdots, p \equiv 0\}$$

($\bar{a}'$ fixed, $p \nmid \bar{a}'$), any two of which intersect in a point $\bar{x}$ divisible by p. So if V contains a primitive ($p \nmid \bar{x}$) point from each subgroup, we have $|V| \geqq p^{\alpha} + p + 1$ and our result follows. Hence we may assume that V does not intersect some $H(\bar{a}')$, where without loss of generality $\bar{a}' = (0, \cdots, 0, 1)$; then V contains no point of type $\bar{x} = \lambda(p\bar{y}_0, 1) \bmod p$ when $p \nmid \lambda$, and hence by (8) no such point for any $\lambda = 1, 2, \cdots$,

$$\bar{x} \in V \Rightarrow \bar{x} = p^{\beta}(\bar{y}_0', y_j) . \qquad (p \nmid \bar{y}_0', 0 \leqq \beta < \alpha) . \tag{17}$$

Now define sets $T(\bar{x})$ as in (14) and denote their union by W,

$$W = \cup\{T(\bar{x}) : \bar{x} \in V\} ,$$

so that $V \subset W \subset S$, and W is the (smallest) set generated by V which satisfies the modified stellar condition (8). Denote

$$W_0 = \{\bar{x}_0 : (\bar{x}_0, x_j) \in W \quad \text{for some} \quad x_j\} . \tag{18}$$

Then by (17), (18), points $\bar{x}_0' p^{\xi}(p \nmid \bar{x}_0')$ of W_0 correspond to points $p^{\xi}(\bar{x}_0', x_j)$ of W and so clearly W_0 satisfies (7) and (8). But V and hence W satisfies every congruence $\bar{l}$ in (9); thus W and hence W_0 satisfies every $\bar{l}$ with $l_j = 0$ for some $\bar{x}_0 = (x_1, \cdots, x_{j-1}) \in W_0$ such that

$$l_1x_1 + \cdots + l_{j-1}x_{j-1} \equiv 0(p^\alpha) \qquad (l_1, \cdots, l_{j-1}, p) = 1 .$$

Thus by our induction hypothesis ($r = j - 1, \alpha \geqq 2$) there are at least $p^\alpha + p$ such $\bar{x}_0 \in W_0$, and

$$|S| \geqq |W| \geqq |W_0| \geqq p^\alpha + p .$$

As our result is already established for $r = \alpha$ (case (i)), this completes the proof of the lemma.

3. Proof of Theorems 2 and 3. Consider the homomorphism η:

$$\mathscr{S} \xrightarrow{\eta} \bar{\mathscr{S}} \cong \mathscr{S}/p^\alpha\mathscr{S} \tag{19}$$

(cf. Cassels [3] for his case $s = 1$); for Theorem 2 we take $\alpha = 1$.

We see easily that if $\mathscr{S} : \mathscr{K} = p^\alpha$ then $p^\alpha\mathscr{S} \subset \mathscr{K}$ and so there is a one-to-one correspondence between all $\mathscr{K}$, $\bar{\mathscr{K}}$ of index p^α in $\mathscr{S}$, $\bar{\mathscr{S}}$ respectively; and any subset V of $\mathscr{S}$ intersects all such $\mathscr{K}$ if and only if $\bar{V}$ intersects all such $\bar{\mathscr{K}}$ (index p^α). If V has the stellar set property this may, however, be lost under η. Since $p^\alpha\bar{\mathscr{S}} = 0$ we have by a result of Prüfer [1] that $\bar{\mathscr{S}}$ is a direct sum of cyclic groups C_i of orders $p^{\beta_i} \leqq p^\alpha$; in fact, $\beta_i = \alpha$ since in all our 3 theorems $\mathscr{S}$ has no element of order $p^\beta (0 < \beta < \alpha)$ and hence $p^{\beta_i}c_i = 0$ implies $\beta_i \geqq \alpha$. Thus

$$\bar{\mathscr{S}} = \sum_{i \in I}^{\oplus} C_i (C_i \cong \langle e : p^\alpha e = 0 \rangle) . \tag{20}$$

Note that all $s \in S$ have infinite period,

$$ms \neq 0 \qquad (s \in S, m = \pm 1, \pm 2, \cdots) \tag{21}$$

since otherwise $|m|s = 0$, $s = (|m| + 1)s \in S$ so $0 = |m|s \in S$ contrary to (2). Now suppose $\bar{0} \in \bar{S}$. Then $p^\alpha g \in S$ so $g, 2g, \cdots, p^\alpha g \in S$, $|S| \geqq p^\alpha$ since otherwise $ig = jg (i < j)$ and $g \in S$ has finite period. It remains therefore to settle the matter when

$$\bar{0} \notin \bar{S} \qquad (\text{i.e., } S \cap p^\alpha\mathscr{S} = \varnothing) . \tag{22}$$

The cases $|I| = 0, 1$ in (20) correspond to groups $\mathscr{S}$ with no, exactly one subgroup of index p^α. In the latter event we have $\bar{0} \in \bar{S}$, a case already settled. If $|I| = 0$ in Theorem 3 then $\mathscr{S} = p^\alpha\mathscr{S}$ and all stellar sets S vacuously satisfy the intersection condition. No stellar set is empty, so we have $s \in S$, $s = p^\alpha s_1$, $s_1 = p^\alpha s_2, \cdots$, and $|S| = \infty$ since otherwise $s_i = s_j$ $(i < j)$ and $s_j \in S$ has finite period, contrary to (21).

The case $|I| \leqq 1$ does not occur for Theorem 2, since here $\mathscr{S}$ has $\geqq 2$ subgroups of index p^α. Hence we may assume

$$|I| \geqq 2 . \tag{23}$$

From (23) it is immediate that $\mathscr{S}$ contains more than one subgroup of index p^α. We consider only Theorem 3 from now on; Theorem 2 will follow by the same reasoning ($\alpha = 1$).

It remains, then, to verify Theorem 3 when (22), (23) hold. Assume now then

$$|S| < \infty , \tag{24}$$

since if $|S| = \infty$ we have nothing to prove. Then if we decompose $\bar{s} = \sum_{s_i}$ in (20) we have $s_i \neq 0$ for some $\bar{s} \in \bar{S}$ for only a finite number of $i \in I$, which we may include in a finite set $i = 1, \cdots, j$ $(2 \leqq j \leqq |I|)$. Then

$$\bar{S} \subset \mathscr{S}^{(0)} \cong \Lambda_0 \bmod p^\alpha \quad (\text{in } j\text{-space } E^j) , \qquad (2 \leqq j) ,$$

$$\bar{\mathscr{S}} = \mathscr{S}^{(0)} \oplus \mathscr{S}^* ,$$

and we may represent any $\bar{x} \in \bar{\mathscr{S}}$ uniquely by

$$\bar{x} = x^{(0)} + x^* = (x_1, \cdots, x_j; x^*) \bmod p^\alpha .$$

The following subgroups $\bar{\mathscr{K}}$ have index p^α in $\bar{\mathscr{S}}$ and hence are intersected by $\bar{S}$:

$$\bar{\mathscr{K}} = \{\bar{x} : l_1 x_1 + \cdots + l_j x_j \equiv 0(p^\alpha)\} \qquad (l_1, \cdots, l_j, p) = 1 ,$$

where $(l_i, p) = 1$ for some i and $l_1, \cdots, l_j$ are fixed for each $\bar{\mathscr{K}}$ (cf. [3, preceding (10)]); we have $p \nmid l_i$ for at least one i and so for each $\bar{x} \in \bar{\mathscr{K}}$, $x_i = -\sum_{j \neq i} l_i^{-1} l_j x_j$. Hence $|\mathscr{K}_0| = p^{\alpha(j-1)}$,

$$\mathscr{S} : \mathscr{K} = \mathscr{S}_0 : \mathscr{K}_0 = p^{\alpha j}/p^{\alpha(j-1)} = p^\alpha .$$

Elements $\bar{s}$ of $\bar{S}$ are of type $\bar{s} = (s_1, \cdots, s_j; 0^*)$; since S is a stellar set the modified property (8) holds for $T = \bar{S}$; also, $0 = (0, \cdots, 0, 0^*) \notin \bar{S}$ and $r = j \geqq 2$ by (22), (23). So we may apply the lemma to find there are at least $p^\alpha + p^{\min(\alpha,2)-1}$ distinct points $(s_1, \cdots, s_j, 0^*)$ in $\bar{S}$; hence

$$|S| \geqq |\bar{S}| \geqq p^\alpha + p^{\min(\alpha,2)-1} ,$$

and our proof of Theorems 2, 3 is complete.

4. Remarks. 1. In our proof of Theorem 3 we utilize the stellar property of S only through its consequence in $\bar{S}$, a condition of type (8) with $T = \bar{S}$ which would clearly follow from imposing condition (8) on S, along with $S \neq \emptyset$. Hence we may make the following extension:

THEOREM 4. *Theorem* 3 *holds for* S *not a stellar set, if* S *satisfies* (8) ($T = S \subset \mathscr{S}$, $\bar{x} \in \mathscr{S}$), and $S \neq \emptyset$.

2. When $\mathscr{S}$ is not abelian, Theorems 1–4 need not hold; e.g., the direct sum $\mathscr{S} = C^\infty \oplus A_5$ of the infinite cyclic group and alternating group of 60 elements has only one subgroup of index 3, $\mathscr{K} = 3C^\infty \oplus A_5$, and $\mathscr{K}$ is intersected by the stellar set of one element,

$$S = \{3 + \text{cycle}\,(123)\} \neq 3g\ .$$

3. In the excluded case $0 \in S$ the least stellar set containing 0 is the periodic part of $\mathscr{S}$, and $|S| \geqq p$ need not follow.

4. When $\mathscr{S} = \Lambda_0(r \geqq 2)$, the set of all $(1, x_2, 0, \cdots, 0)$, $(px_1, 1, 0, \cdots, 0) \bmod p^\alpha$ is a stellar set of $p^\alpha + p^{\alpha-1}$ elements intersecting all congruences (9) mod p^α. So our bounds are best possible, for the lemma, when $\alpha = 1, 2$. ($r \geqq 2$).

5. In Theorem 3 we must exclude elements of order $p^\beta(\beta < \alpha)$. For consider ,e.g., $\mathscr{S} = C^\infty \oplus C^{(p)}$ (any α). Here the bound is $p^\alpha + 1$.

6. Let $\alpha \geqq 2$, S be a stellar set in Euclidean n-space $\{\bar{x} = (x_1, \cdots, x_n)\}$ with fewer than $p^\alpha + p$ elements, and no element $p^\alpha \bar{x}$. Then there is a sublattice of the fundamental lattice of determinant p^α (see [2], p. 10) which is not intersected by S.

7. Our condition $(A)''S$ intersects all subgroups of index n'' is equivalent to $(B)'' \cdots$ index $d\colon d\,|\,n''$ though weaker than $(C)'' \cdots$ index $m\colon m < n''$. The latter remark follows from the example $S = \{(4,1), (2,1), (2,0), (1,0)\}$ in $\mathscr{S} = C^\infty \oplus C^{(2)}$ ($n = 4$). For the former prove first for $d = n/p$ and then iterate: if $\mathscr{S} : \mathscr{H} = n/p$ $(p\,|\,n)$ and (A) holds then $\mathscr{H} \neq p\mathscr{H}$, there exist $\mathscr{M}$ in $\mathscr{H}$ with $\mathscr{H} : \mathscr{M} = p$ so $\mathscr{S} : \mathscr{M} = n$, $\mathscr{H} \cap S \neq \emptyset$.

8. Theorem 3 does not hold for all $n = 1, 2, \cdots$. Mr. George M. Bergman of Cambridge, Mass. has kindly furnished me with a set of counterexamples for $\mathscr{S} = C^\infty \oplus C^\infty$, which includes a stellar set S of 76 elements that intersects every subgroup of index 77.

9. Finally, we should like to acknowledge here some parallel though independent work of Mr. Bergman who in unpublished cor-

respondence proves a simpler version of Theorem 4, obtaining a slightly lower bound (p^α rather than $p^\alpha + p$, 1). His proof is in essence similar to ours, except there is no induction step: a homomorphism η (19) reduces the problem to Rogers' case $\mathscr{S} = \Lambda_0$, and a version of our lemma is proved by arguments resembling ours for $\alpha = 1$ or $r \geqq \alpha$, Mr. Bergman in effect considering congruences (9) with $l_1 = 1$ to obtain his bound p^α for (10) for all r, α, without induction. We thank Mr. Bergman for the material communicated; among other things it helped remind us to include Theorem 4. We thank him also for welcome suggestions concerning our final draft.

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