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**ON SUBGROUPS OF FIXED INDEX**

G. K. WHITE

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GEORGE K. WHITE

If  $k \in \mathcal{K}$ , where  $\mathcal{K}$  is a subgroup of a group  $\mathcal{S}$ , then closure implies  $k^2, k^3, \dots, \in \mathcal{K}$ . Nonempty subsets  $S \subset \mathcal{S}$  with the inverse property  $s^m \in S$  implies  $s, s^2, \dots, s^m \in S$  ( $m = 1, 2, \dots$ ) will be called *stellar sets*. Let  $p^\alpha$  be a fixed prime power. If a stellar set  $S$  of an abelian group  $\mathcal{S}$  intersects every subgroup  $\mathcal{H}$  of index  $p^\alpha$  in  $\mathcal{S}$ , and  $0 \notin S$ , then the cardinal  $|S|$  of  $S$  is bounded below by  $p^\alpha$  (Theorem 3), when  $\mathcal{S}$  satisfies a mild condition.

Hence for instance a subset  $S$  of euclidean  $n$ -space  $E_n$  intersecting all sublattices of determinant  $p^\alpha$  of the fundamental lattice will have at least  $p^\alpha$  elements, and more if no element is divisible by  $p^\alpha$ .

Henceforth  $\mathcal{S}$  will always be an additive abelian group, so a *stellar set* will be one with

$$(1) \quad \begin{aligned} & \emptyset \neq S \subset \mathcal{S} \\ & mg \in S \Rightarrow g, 2g, \dots, mg \in S (g \in \mathcal{S}, m = 1, 2, \dots) . \end{aligned}$$

Examples of stellar sets are  $\mathcal{S}$  itself, and its *periodic part* [5, p. 137]; and a *star set* [7] is a symmetric stellar set. There are stellar sets of one element  $s$ , i.e., those  $s$  for which  $s = mg (m = 1, 2, \dots)$  implies  $m = 1$ . Now let  $p$  be a fixed prime, and suppose  $S$  intersects every subgroup  $\mathcal{H}$  of  $\mathcal{S}$  of index  $p$ . Suppose also

$$(2) \quad 0 \notin S$$

(if  $0 \in S$  the intersection property is redundant). Then we can say the following (in this paper we denote  $|A| =$  cardinal of  $A$ ,  $mA = \{ma; a \in A\}$ , for any set  $A$  and integer  $m$ ):

**THEOREM 1.** *Let  $p$  be a fixed prime,  $\mathcal{S}$  an abelian group, and  $S$  a stellar set with  $0 \notin S$  which intersects all subgroups  $\mathcal{H}$  of index  $\mathcal{S} : \mathcal{H} = p$ . Then*

$$(3) \quad |S| \geq p .$$

When  $S \cap p\mathcal{S} = \emptyset$  we have  $|S| > p$ .

A similar result holds for ordinary sets  $T$ :

**THEOREM 2.** *Suppose  $p$  is a fixed prime,  $\mathcal{S}$  is an abelian group with more than one subgroup of index  $p$ , and  $T$  is any subset of  $\mathcal{S}$  with*

$$(4) \quad T \cap p\mathcal{S} = \emptyset$$

which intersects all subgroups  $\mathcal{K}$  of index  $\mathcal{S} : \mathcal{K} = p$ . Then

$$(5) \quad |T| \geq p + 1.$$

When  $\mathcal{S}$  is the fundamental lattice  $A_0$  [2, 4] in  $r$ -space  $E_r$  of all points with integral coordinates, Theorems 1 and 2 are immediate using Rogers' proof of his Theorem 1 [7] on starsets, the small adjustment needed being clear. He states his theorem with a slightly stronger hypothesis equivalent to " $S$  intersects all subgroups of index  $\leq p$ ", and for this more stringent requirement Cassels [3], replacing  $p$  by  $n$ , has made elegant use of a generalization of Bertrand's postulate due to Sylvester [9] and Schur [8] to show  $|S| \geq n$  for  $n = 1, 2, \dots$  and any stellar set  $S$  of an abelian  $\mathcal{S}$  with no periodic part. For  $n = p^\alpha$  a prime power we shall extend this as follows:

**THEOREM 3.** *Suppose that  $n = p^\alpha$  is fixed,  $\mathcal{S}$  is an abelian group containing no element of order  $p^\beta$  when  $1 < p^\beta < p^\alpha$ , and that  $S$  is a stellar set with  $0 \notin S$  which intersects all subgroups  $\mathcal{K}$  of index  $\mathcal{S} : \mathcal{K} = p^\alpha$ . Then*

$$(6) \quad |S| \geq p^\alpha.$$

When  $S \cap p^\alpha\mathcal{S} = \emptyset$ , we have

$$|S| \geq p^\alpha + \begin{cases} p & \text{if } \alpha > 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

Note the requirement "contains at least one subgroup of index  $p^\alpha$ " is a natural one, but it is an unneeded restriction on  $S$ . Note also that Theorem 1 is an immediate consequence of Theorem 3.

2. **A lemma.** We find it useful, for Rogers' case  $\mathcal{S} = A_0 \subset E^r$ , to restate Theorem 3 in altered form. We denote  $\bar{x} = (x_1, \dots, x_r)$  so that

$$A_0 = \{\bar{x} : \text{all the } x_i \text{ are integers, } i = 1, \dots, r\},$$

and  $\mathcal{S} = A_0$  is isomorphic to a direct sum of  $r$  infinite cyclic groups. When  $\bar{x} \in A_0$  we define  $p|\bar{x}$  to mean  $p|x_1, \dots, p|x_r$ , and

$$\|x\|_p = \max \{\alpha : p^\alpha | \bar{x}\}.$$

Let  $T$  be any subset of  $A_0$  satisfying

$$(7) \quad p^\alpha A_0 \cap T = \emptyset \quad (T \subset A_0),$$

and a modified stellar condition

$$(8) \quad \begin{cases} p^\beta \bar{x} \in T \text{ implies } \bar{x}, 2\bar{x}, \dots, p^\beta \bar{x} \in T \\ (1 \leq \beta \leq \alpha, p^\alpha \text{ fixed}), \end{cases}$$

and consider congruences

$$(9) \quad \bar{l} \cdot \bar{x} = l_1 x_1 + \dots + l_r x_r \equiv 0(p^\alpha) (\bar{l} \in A_0, p \nmid \bar{l}).$$

LEMMA. *If  $T \subset A_0$  satisfies (7) and (8),  $r \geq 2$  and the congruence (9) has for each  $\bar{l}$  a solution  $\bar{x} \in T$ , then  $T$  contains at least  $p^\alpha + p^{\min(\alpha, 2)-1}$  distinct elements mod  $p^\alpha$ ,*

$$(10) \quad |T \text{ mod } p^\alpha| \geq p^\alpha + \begin{cases} p & \text{if } \alpha > 1, \\ 1 & \text{if } \alpha = 1. \end{cases}$$

*Proof.* We consider two cases, (i)  $\alpha = 1$  or  $r \leq \alpha$ , and (ii)  $r > \alpha \geq 2$ . For the first case, a simple counting argument will suffice. Define

$$(11) \quad \theta(i, \alpha) = \frac{p^{(i-1)(\alpha-1)}(p^i - 1)}{p - 1}.$$

Then there are exactly

$$\sum_{k=1}^{k=r} p^{(\alpha-1)(k-1) + \alpha(r-k)} = \theta(r, \alpha)$$

distinct congruences (9), representable by

$$\bar{l} = (pm_1, \dots, pm_{k-1}, 1, l_{k+1}, \dots, l_r).$$

If  $\bar{y} \equiv b\bar{x} \text{ mod } p^\alpha$  then clearly  $\bar{y}$  satisfies every congruence  $\bar{x}$  does, and hence we may construct a subset  $V$  of  $T$  which likewise satisfies every congruence (9), and also

$$(12) \quad \begin{cases} \bar{x} \in V, \bar{y} \in V, \bar{y} \equiv b\bar{x} \text{ mod } p^\alpha \Rightarrow \bar{y} = \bar{x}, \\ \bar{x} \in V \Rightarrow \bar{x} \text{ satisfies some congruence (9)}. \end{cases}$$

Any  $\bar{x} \in V$  may be expressed as

$$\bar{x} = \bar{x}' p^\xi (p \nmid \bar{x}'; 0 \leq \xi = \|\bar{x}\|_p < \alpha)$$

by (7), since  $V \subset T$ . A fixed  $\bar{x} \in V$  obeys (9) for at least one  $\bar{l}$  and in fact for precisely those  $\bar{l}$  satisfying  $\bar{l} \cdot \bar{x}' \equiv 0(p^{\alpha-\xi})$ ; these correspond to exactly  $p^\xi \theta(r-1, \alpha)$  congruences (consider, e.g.,  $\bar{x}' = (1, 0, \dots, 0)$ ). Hence, counting over the  $\theta(r, \alpha)$  congruences (9), we get

$$(13) \quad \theta(r, \alpha) \leq \sum_{\bar{x} \in V} p^{\|\bar{x}\|_p} \theta(r-1, \alpha).$$

Now  $\bar{x} \in V$  obeys (8), since  $V \subset T$ . Hence to each  $\bar{x} = \bar{x}' p^\xi$  in  $V$  there correspond  $p^\xi$  elements

$$(14) \quad T(\bar{x}) = \{\lambda \bar{x}' : \lambda = 1, \dots, p^\epsilon\} \subset T \quad (\bar{x} \in V).$$

Moreover,

$$(15) \quad \bar{x}_1 \neq \bar{x}_2 \text{ implies } T(\bar{x}_1) \cap T(\bar{x}_2) = \emptyset \quad (\bar{x}_1, \bar{x}_2 \in V),$$

for otherwise  $\lambda_1 \bar{x}_1 = \lambda_2 \bar{x}_2$ ,  $\lambda_i = \lambda'_i p^{0i} (p \nmid \lambda'_i)$ , without loss of generality  $\theta = \theta_1 - \xi_1 - (\theta_2 - \xi_2) \geq 0$ , and  $\lambda'_2 \bar{x}_2 = \lambda'_1 p^\theta \bar{x}_1$ ,  $\bar{x}_2 \equiv (\lambda'_2)^{-1} \lambda'_1 p^\theta \bar{x}_1 \pmod{p^\alpha}$ ,  $\bar{x}_2 = \bar{x}_1$  by (12). Thus by (13), (15),

$$\begin{aligned} |T| &\geq \sum_{\bar{x} \in V} p^{|\bar{x}|} \geq \theta(r, \alpha) / \theta(r - 1, \alpha) \\ &= p^\alpha + \frac{p^{\alpha-1}(p - 1)}{p^{r-1} - 1}. \quad (r \geq 2) \end{aligned}$$

If  $\alpha = 1$  we have  $|T| \geq p + (p - 1)(p^{r-1} - 1)^{-1} > p$ , so  $|T| \geq p + 1$ ; if  $r \leq \alpha > 1$  then

$$|T| - p^\alpha \geq p^{\alpha-1}(p - 1)(p^{\alpha-1} - 1)^{-1} > p - 1,$$

$|T| - p^\alpha \geq p$ , and case (i) is verified.

For our second case  $r > \alpha \geq 2$  we employ induction on  $r$ . Let  $r = j$ , define  $V \subset T$  as in case (i), and denote

$$(16) \quad \bar{x} = (x_1, \dots, x_{j-1}, x_j) = (\bar{x}_0, x_j).$$

There are  $p^{j-1} + \dots + p + 1 \geq p^\alpha + p + 1$  subgroups

$$H(\bar{a}') = \{\lambda \bar{a}' \pmod{p} : \lambda = 1, \dots, p \equiv 0\}$$

( $\bar{a}'$  fixed,  $p \nmid \bar{a}'$ ), any two of which intersect in a point  $\bar{x}$  divisible by  $p$ . So if  $V$  contains a primitive ( $p \nmid \bar{x}$ ) point from each subgroup, we have  $|V| \geq p^\alpha + p + 1$  and our result follows. Hence we may assume that  $V$  does not intersect some  $H(\bar{a}')$ , where without loss of generality  $\bar{a}' = (0, \dots, 0, 1)$ ; then  $V$  contains no point of type  $\bar{x} = \lambda(p\bar{y}_0, 1) \pmod{p}$  when  $p \nmid \lambda$ , and hence by (8) no such point for any  $\lambda = 1, 2, \dots$ ,

$$(17) \quad \bar{x} \in V \Rightarrow \bar{x} = p^\beta(\bar{y}'_0, y_j). \quad (p \nmid \bar{y}'_0, 0 \leq \beta < \alpha).$$

Now define sets  $T(\bar{x})$  as in (14) and denote their union by  $W$ ,

$$W = \cup \{T(\bar{x}) : \bar{x} \in V\},$$

so that  $V \subset W \subset S$ , and  $W$  is the (smallest) set generated by  $V$  which satisfies the modified stellar condition (8). Denote

$$(18) \quad W_0 = \{\bar{x}_0 : (\bar{x}_0, x_j) \in W \text{ for some } x_j\}.$$

Then by (17), (18), points  $\bar{x}'_0 p^\epsilon (p \nmid \bar{x}'_0)$  of  $W_0$  correspond to points  $p^\epsilon(\bar{x}'_0, x_j)$  of  $W$  and so clearly  $W_0$  satisfies (7) and (8). But  $V$  and

hence  $W$  satisfies every congruence  $\bar{l}$  in (9); thus  $W$  and hence  $W_0$  satisfies every  $\bar{l}$  with  $l_j = 0$  for some  $\bar{x}_0 = (x_1, \dots, x_{j-1}) \in W_0$  such that

$$l_1x_1 + \dots + l_{j-1}x_{j-1} \equiv 0(p^\alpha) \quad (l_1, \dots, l_{j-1}, p) = 1.$$

Thus by our induction hypothesis ( $r = j - 1, \alpha \geq 2$ ) there are at least  $p^\alpha + p$  such  $\bar{x}_0 \in W_0$ , and

$$|S| \geq |W| \geq |W_0| \geq p^\alpha + p.$$

As our result is already established for  $r = \alpha$  (case (i)), this completes the proof of the lemma.

3. Proof of Theorems 2 and 3. Consider the homomorphism  $\eta$ :

$$(19) \quad \mathcal{S} \xrightarrow{\eta} \bar{\mathcal{S}} \cong \mathcal{S}/p^\alpha \mathcal{S}$$

(cf. Cassels [3] for his case  $s = 1$ ); for Theorem 2 we take  $\alpha = 1$ .

We see easily that if  $\mathcal{S} : \mathcal{K} = p^\alpha$  then  $p^\alpha \mathcal{S} \subset \mathcal{K}$  and so there is a one-to-one correspondence between all  $\mathcal{K}, \bar{\mathcal{K}}$  of index  $p^\alpha$  in  $\mathcal{S}, \bar{\mathcal{S}}$  respectively; and any subset  $V$  of  $\mathcal{S}$  intersects all such  $\mathcal{K}$  if and only if  $\bar{V}$  intersects all such  $\bar{\mathcal{K}}$  (index  $p^\alpha$ ). If  $V$  has the stellar set property this may, however, be lost under  $\eta$ . Since  $p^\alpha \mathcal{S} = 0$  we have by a result of Prüfer [1] that  $\bar{\mathcal{S}}$  is a direct sum of cyclic groups  $C_i$  of orders  $p^{\beta_i} \leq p^\alpha$ ; in fact,  $\beta_i = \alpha$  since in all our 3 theorems  $\mathcal{S}$  has no element of order  $p^\beta (0 < \beta < \alpha)$  and hence  $p^{\beta_i} c_i = 0$  implies  $\beta_i \geq \alpha$ . Thus

$$(20) \quad \bar{\mathcal{S}} = \sum_{i \in I}^{\oplus} C_i (C_i \cong \langle e : p^\alpha e = 0 \rangle).$$

Note that all  $s \in S$  have infinite period,

$$(21) \quad ms \neq 0 \quad (s \in S, m = \pm 1, \pm 2, \dots)$$

since otherwise  $|m|s = 0, s = (|m| + 1)s \in S$  so  $0 = |m|s \in S$  contrary to (2). Now suppose  $\bar{0} \in \bar{S}$ . Then  $p^\alpha g \in S$  so  $g, 2g, \dots, p^\alpha g \in S, |S| \geq p^\alpha$  since otherwise  $ig = jg (i < j)$  and  $g \in S$  has finite period. It remains therefore to settle the matter when

$$(22) \quad \bar{0} \notin \bar{S} \quad (\text{i.e., } S \cap p^\alpha \mathcal{S} = \emptyset).$$

The cases  $|I| = 0, 1$  in (20) correspond to groups  $\mathcal{S}$  with no, exactly one subgroup of index  $p^\alpha$ . In the latter event we have  $\bar{0} \in \bar{S}$ , a case already settled. If  $|I| = 0$  in Theorem 3 then  $\mathcal{S} = p^\alpha \mathcal{S}$  and all stellar sets  $S$  vacuously satisfy the intersection condition. No stellar set is empty, so we have  $s \in S, s = p^\alpha s_1, s_1 = p^\alpha s_2, \dots$ , and  $|S| = \infty$  since otherwise  $s_i = s_j (i < j)$  and  $s_j \in S$  has finite period, contrary to (21).

The case  $|I| \leq 1$  does not occur for Theorem 2, since here  $\mathcal{S}$  has  $\geq 2$  subgroups of index  $p^\alpha$ . Hence we may assume

$$(23) \quad |I| \geq 2.$$

From (23) it is immediate that  $\mathcal{S}$  contains more than one subgroup of index  $p^\alpha$ . We consider only Theorem 3 from now on; Theorem 2 will follow by the same reasoning ( $\alpha = 1$ ).

It remains, then, to verify Theorem 3 when (22), (23) hold. Assume now then

$$(24) \quad |S| < \infty,$$

since if  $|S| = \infty$  we have nothing to prove. Then if we decompose  $\bar{s} = \sum_{s_i}$  in (20) we have  $s_i \neq 0$  for some  $\bar{s} \in \bar{S}$  for only a finite number of  $i \in I$ , which we may include in a finite set  $i = 1, \dots, j$  ( $2 \leq j \leq |I|$ ). Then

$$\bar{S} \subset \mathcal{S}^{(0)} \cong A_0 \bmod p^\alpha \quad (\text{in } j\text{-space } E^j), \quad (2 \leq j),$$

$$\bar{\mathcal{S}} = \mathcal{S}^{(0)} \oplus \mathcal{S}^*,$$

and we may represent any  $\bar{x} \in \bar{\mathcal{S}}$  uniquely by

$$\bar{x} = x^{(0)} + x^* = (x_1, \dots, x_j; x^*) \bmod p^\alpha.$$

The following subgroups  $\bar{\mathcal{K}}$  have index  $p^\alpha$  in  $\bar{\mathcal{S}}$  and hence are intersected by  $\bar{S}$ :

$$\bar{\mathcal{K}} = \{\bar{x}: l_1 x_1 + \dots + l_j x_j \equiv 0(p^\alpha)\} \quad (l_1, \dots, l_j, p) = 1,$$

where  $(l_i, p) = 1$  for some  $i$  and  $l_1, \dots, l_j$  are fixed for each  $\bar{\mathcal{K}}$  (cf. [3, preceding (10)]); we have  $p \nmid l_i$  for at least one  $i$  and so for each  $\bar{x} \in \bar{\mathcal{K}}$ ,  $x_i = -\sum_{j \neq i} l_i^{-1} l_j x_j$ . Hence  $|\bar{\mathcal{K}}_0| = p^{\alpha(j-1)}$ ,

$$\mathcal{S} : \bar{\mathcal{K}} = \mathcal{S}_0 : \bar{\mathcal{K}}_0 = p^{\alpha j} / p^{\alpha(j-1)} = p^\alpha.$$

Elements  $\bar{s}$  of  $\bar{S}$  are of type  $\bar{s} = (s_1, \dots, s_j; 0^*)$ ; since  $S$  is a stellar set the modified property (8) holds for  $T = \bar{S}$ ; also,  $0 = (0, \dots, 0, 0^*) \notin \bar{S}$  and  $r = j \geq 2$  by (22), (23). So we may apply the lemma to find there are at least  $p^\alpha + p^{\min(\alpha, 2)-1}$  distinct points  $(s_1, \dots, s_j, 0^*)$  in  $\bar{S}$ ; hence

$$|S| \geq |\bar{S}| \geq p^\alpha + p^{\min(\alpha, 2)-1},$$

and our proof of Theorems 2, 3 is complete.

4. Remarks. 1. In our proof of Theorem 3 we utilize the stellar property of  $S$  only through its consequence in  $\bar{S}$ , a condition of type (8) with  $T = \bar{S}$  which would clearly follow from imposing

condition (8) on  $S$ , along with  $S \neq \emptyset$ . Hence we may make the following extension:

**THEOREM 4.** *Theorem 3 holds for  $S$  not a stellar set, if  $S$  satisfies (8) ( $T = S \subset \mathcal{S}$ ,  $\bar{x} \in \mathcal{S}$ ), and  $S \neq \emptyset$ .*

2. When  $\mathcal{S}$  is not abelian, Theorems 1-4 need not hold; e.g., the direct sum  $\mathcal{S} = C^\infty \oplus A_5$  of the infinite cyclic group and alternating group of 60 elements has only one subgroup of index 3,  $\mathcal{H} = 3C^\infty \oplus A_5$ , and  $\mathcal{H}$  is intersected by the stellar set of one element,

$$S = \{3 + \text{cycle } (123)\} \neq 3g .$$

3. In the excluded case  $0 \in S$  the least stellar set containing 0 is the periodic part of  $\mathcal{S}$ , and  $|S| \geq p$  need not follow.

4. When  $\mathcal{S} = A_0(r \geq 2)$ , the set of all  $(1, x_2, 0, \dots, 0), (px_1, 1, 0, \dots, 0) \pmod{p^\alpha}$  is a stellar set of  $p^\alpha + p^{\alpha-1}$  elements intersecting all congruences (9) mod  $p^\alpha$ . So our bounds are best possible, for the lemma, when  $\alpha = 1, 2$ . ( $r \geq 2$ ).

5. In Theorem 3 we must exclude elements of order  $p^\beta (\beta < \alpha)$ . For consider ,e.g.,  $\mathcal{S} = C^\infty \oplus C^{(p)}$  (any  $\alpha$ ). Here the bound is  $p^\alpha + 1$ .

6. Let  $\alpha \geq 2$ ,  $S$  be a stellar set in Euclidean  $n$ -space  $\{\bar{x} = (x_1, \dots, x_n)\}$  with fewer than  $p^\alpha + p$  elements, and no element  $p^\alpha \bar{x}$ . Then there is a sublattice of the fundamental lattice of determinant  $p^\alpha$  (see [2], p. 10) which is not intersected by  $S$ .

7. Our condition (A)'' $S$  intersects all subgroups of index  $n''$  is equivalent to (B)''  $\dots$  index  $d: d|n''$  though weaker than (C)''  $\dots$  index  $m: m < n''$ . The latter remark follows from the example  $S = \{(4,1), (2,1), (2,0), (1,0)\}$  in  $\mathcal{S} = C^\infty \oplus C^{(2)}$  ( $n = 4$ ). For the former prove first for  $d = n/p$  and then iterate: if  $\mathcal{S} : \mathcal{H} = n/p$  ( $p|n$ ) and (A) holds then  $\mathcal{H} \neq p\mathcal{H}$ , there exist  $\mathcal{M}$  in  $\mathcal{H}$  with  $\mathcal{H} : \mathcal{M} = p$  so  $\mathcal{S} : \mathcal{M} = n$ ,  $\mathcal{H} \cap S \neq \emptyset$ .

8. Theorem 3 does not hold for all  $n = 1, 2, \dots$ . Mr. George M. Bergman of Cambridge, Mass. has kindly furnished me with a set of counterexamples for  $\mathcal{S} = C^\infty \oplus C^\infty$ , which includes a stellar set  $S$  of 76 elements that intersects every subgroup of index 77.

9. Finally, we should like to acknowledge here some parallel though independent work of Mr. Bergman who in unpublished cor-



respondence proves a simpler version of Theorem 4, obtaining a slightly lower bound ( $p^\alpha$  rather than  $p^\alpha + p, 1$ ). His proof is in essence similar to ours, except there is no induction step: a homomorphism  $\eta$  (19) reduces the problem to Rogers' case  $\mathcal{S} = \Lambda_0$ , and a version of our lemma is proved by arguments resembling ours for  $\alpha = 1$  or  $r \geq \alpha$ , Mr. Bergman in effect considering congruences (9) with  $l_1 = 1$  to obtain his bound  $p^\alpha$  for (10) for all  $r, \alpha$ , without induction. We thank Mr. Bergman for the material communicated; among other things it helped remind us to include Theorem 4. We thank him also for welcome suggestions concerning our final draft.

#### REFERENCES

1. J. L. Brenner, *The decomposition theorem for abelian groups* Bull. Amer. Math. Soc. **47** (1941), 116-117.
2. J. W. S. Cassels, *An introduction to the geometry of numbers* Grund d. Mat. Wiss **99** (Berlin, 1959).
3. ———, *On the subgroups of infinite abelian groups*, J. London Math. Soc. **33** (1958), 281-284.
4. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford, 1954.
5. A. C. Kurosh, *The theory of groups I* (Trans. K. A. Hirsch), Chelsea, New York, 1955.
6. H. Prüfer, *Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen*, Math. Zeit. **17** (1923), 35-61.
7. C. A. Rogers, *The number of lattice points in a star body* J. London Math. Soc. **26** (1951), 307-310.
8. I. Schur, *Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen* I, II Sitzungsberichte der Preuss. Akad. Wiss. (1929), 125-136, 370-391.
9. J. Sylvester, *On arithmetical series*, Mess of Math. **21** (1892), 1-19, 87-120. (Reprinted in Collected Works IV, 687-731.)

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