

Pacific Journal of Mathematics

**FINITE GROUPS IN WHICH EVERY ELEMENT IS
CONJUGATE TO ITS INVERSE**

J. LENNART (JOHN) BERGGREN

FINITE GROUPS IN WHICH EVERY ELEMENT IS CONJUGATE TO ITS INVERSE

J. L. BERGGREN

Let \mathfrak{S} denote the class of all finite groups all of whose irreducible characters over \mathbb{C} (the complex numbers) are real. It is easy to verify, but important to observe, that this condition is equivalent to the condition that every element of the group is conjugate to its inverse (under an inner automorphism). Since $S_n \in \mathfrak{S}$, where S_n denotes the symmetric group on n letters, any finite group may be embedded in a group in \mathfrak{S} . The goal of §1 will be to show that the two-Sylow subgroups of S_n also are in \mathfrak{S} , and, if A_n denotes the alternating group on n letters, that $A_n \in \mathfrak{S}$ if and only if $n \in \{1, 2, 5, 6, 10, 14\}$. The result on the two-Sylow subgroups of S_n will be used to show that any finite two-group is embeddable in a two-group in \mathfrak{S} . G. A. Miller has studied a class of groups related to those in \mathfrak{S} . The main theorem of §2 gives a more intuitive characterization of the class of groups investigated by Miller, a consequence of which is a necessary and sufficient condition for a group in this class to be a member of \mathfrak{S} .

NOTATION. Throughout this paper we shall adhere to the notation of M. Hall [2], with these exceptions: $Fr(G)$ will denote the Frattini subgroup of the group G and GwH will denote the wreath product of G with H . The word "group" will denote a finite group. If n, n_1, \dots, n_k are positive integers then the symbol $n = [n_1, \dots, n_k]$ will mean $n = n_1 + \dots + n_k$.

1. Burnside [1] observed that any nonidentity group in \mathfrak{S} has even order. So, in investigating such groups one might start with those groups in \mathfrak{S} that are two-groups. However, this section will show that, without additional hypotheses, there can be no general structure theorems for these.

THEOREM 1.1. *If $A = \langle a \rangle$ has order 2 and if $B \in \mathfrak{S}$ then $BwA \in \mathfrak{S}$.*

Proof. Since $(B, B^a) = 1$ and $B \in \mathfrak{S}$, elements of $B \times B^a$ are immediately conjugate to their inverses. It suffices, therefore, to consider elements in the coset $a(B \times B^a)$. These elements are of the form abc^a , where $b, c \in B$. But $abc^a = ac^ab = cab$ and $(cab)^c = abc$. Hence it suffices to consider elements of aB , say ab . Pick $x \in B$,

$b^x = b^{-1}$. Let $h = xx^a a$. Since $(a, xx^a) = 1 = (b, x^a)$, $(ab)^h = ab^{x^a} = b^{-1}a = (ab)^{-1}$. Hence $BwA \in \mathfrak{S}$.

COROLLARY 1.1. *If T_n denotes a two-Sylow subgroup of S_n then $T_n \in \mathfrak{S}$.*

Proof. If $n = 2^m$ ($m = 1, 2, \dots$) then $T^n = (\dots (CwC)wC)w \dots)wC$ is just the iterated wreath product of C m times, where C is the cyclic group of order 2, and the result follows from Theorem 1.1. But for arbitrary n , T_n is the direct product of groups of the above form. Since $A \times B \in \mathfrak{S}$ whenever A and B are in \mathfrak{S} , $T_n \in \mathfrak{S}$ for all n .

COROLLARY 1.2. *If G is any two-group then there exists a two-group $H \in \mathfrak{S}$ and a monomorphism $\tau : G \rightarrow H$.*

Proof. This follows from the Cayley Theorem and Corollary 1.1.

REMARK. Wreath-products are extremely useful in constructing counterexamples to conjectures about two-groups in \mathfrak{S} . For example: Let A and B be elementary abelian groups, of orders 2^m and 2^n respectively. Let $G = AwB$. Since any element of AwB may be written as the product of two involutions it is immediate that $G \in \mathfrak{S}$. But G has exponent 4 and nilpotence class at least n . So it is in general not possible to bound the class of $G \in \mathfrak{S}$ by bounding its exponent (except in the trivial case where the exponent is 2).

The only nilpotent groups in \mathfrak{S} are two-groups. For $G = \prod_{i=1}^n H_i \in \mathfrak{S}$ if and only if $H_i \in \mathfrak{S}$ for $i = 1, \dots, n$. It follows that, since any group in \mathfrak{S} has even order, if G is any nilpotent group in \mathfrak{S} then G is a two-group. Also, from Theorem 10.5.3 of [2], it follows that if G is super-solvable and if $G \in \mathfrak{S}$ then any two-Sylow subgroup of G is in \mathfrak{S} (for if $G \in \mathfrak{S}$ then any homomorphic image of G is also in \mathfrak{S}).

Having shown that the two-Sylow subgroups of S_n are in \mathfrak{S} for all n we shall now prove that $A_n \in \mathfrak{S}$ if and only if $n \in \{1, 2, 5, 6, 10, 14\}$. In particular we conclude that \mathfrak{S} contains non-abelian simple groups.

The following terminology will be convenient in what follows. If $g \in A_n$ and $g = (u_1, \dots, u_r) \dots (v_1, \dots, v_s)$ is an expression for g as a product of disjoint cycles then, setting $b = (u_2, u_r)(u_3, u_{r-1}) \dots (v_2, v_s) \dots$ yields $g^b = g^{-1}$. The above b will be called a *standard conjugator* of g .

THEOREM 1.2. *The alternating group $A_n \in \mathfrak{S}$ if and only if $n \in \{1, 2, 5, 6, 10, 14\}$.*

Proof. If $[n_1, \dots, n_k]$ is a partition of n into distinct odd integers such that the number of $n_i \equiv 3(4)$ is odd then $A_n \notin \mathfrak{S}$. For, let g be

any element of A_n corresponding to this partition. By Theorem 11.1.5 of [4], $[S_n: C_{S_n}(g)] > [A_n: C_{A_n}(g)]$. If $C_{S_n}(g) \not\subseteq A_n$ then $[S_n: C_{S_n}(g)] = [A_n: C_{S_n}(g) \cap A_n] = [A_n: C_{A_n}(g)]$, which is a contradiction. Thus, $C_{S_n}(g) \subseteq A_n$. Since the number of $n_i \equiv 3(4)$ is odd the standard conjugator α of g is in $S_n - A_n$. If $g^\beta = g^{-1}$ for some $\beta \in A_n$ then $\alpha\beta \in C_{S_n}(g) \cap (S_n - A_n)$, which is a contradiction.

Next, notice that there exists a partition $[n_1, \dots, n_k]$ of n into distinct odd integers such that the number of $n_i \equiv 3(4)$ is odd unless $n = 1, 2, 5, 6, 10, 14$. If $n = 4k$ then $[4k - 3, 3,]$ is such a partition. If $n = 4k + 3$ then $[4k + 3]$ is such a partition. If $n = 4k + 1$ then $n = [4k - 3, 3, 1]$ is such a partition provided $k > 1$ and, if $n = 4k + 2, k > 3$, then $[5, 1, 3, 4(k - 1) - 3]$ is such a partition. When $n = 1, 2, 5, 6, 10, 14$ every partition of n either contains an even integer, a repeated integer, or else the number of $n_i \equiv 3(4)$ is even.

Now if there exists an even integer or a repeated integer in a partition of n then any corresponding element $g \in S_n$ either is not in A_n or $[S_n: C_{S_n}(g)] = [A_n: C_{A_n}(g)]$, by Theorem 11.1.5 of [4]. In particular, g is conjugate to g^{-1} in A_n .

The remaining elements g of A_n correspond to partitions of n into distinct odd integers where the number of $n_i \equiv 3(4)$ is even. But this implies any standard conjugator of g is in A_n , so $g^\alpha = g^{-1}$ where $\alpha \in A_n$.

2. In [3] G. A. Miller examined the structure of those n -generator groups $M_n = \langle t_1, \dots, t_n \rangle$ ($n > 1$) such that $|t_i| > 2$ and $t_j^{-1}t_it_j = t_i^{-1}$, $1 \leq i \neq j \leq n$. We shall refer to M_n as the Miller group on n generators.

Let $M_n = \langle t_1, \dots, t_n \rangle$ and $S = \{t_1, \dots, t_n\}$. If $u, v \in S$ a short calculation shows that $|u| = 4$ and $u^2 = v^2$. Clearly, then, $|M_n^{(1)}| = 2$ and every element of M_n may be written uniquely apart from the order of the factors as $w \dots z \cdot t_1^{2i}$, where w, \dots, z and distinct elements of S and $i \in \{0, 1\}$. The main theorem of this section will show that M_n is built out of quaternion, dihedral, and cyclic groups. As a corollary we shall classify those n for which $M_n \in \mathfrak{C}$.

LEMMA 2.1. *Let $g = s_1 \dots s_m \cdot t_1^{2i} \in M_n$ (where each s_k is a t_j and $s_j = s_k$ if and only if $j = k$). Then $|g| = 4$ if and only if $m(m + 1) \not\equiv 0(4)$. If, also, $h = s'_1 \dots s'_n \cdot t_1^{2k}$ (with the same conditions on the s'_i) then $(g, h) = t_1^{2\alpha}$, where $\alpha = mn - c(g, h)$ and $c(g, h) = |\{(i, k) | s_i = s'_k\}|$.*

Proof. An easy calculation shows $g^2 = c^\beta$, where $\beta = m(m + 1)/2$ and $c = t_1^2$. As $|c| = 2$ the conclusion to the first part follows. Since all commutators are central and any commutator is equal to 1 or t_1^2

we have

$$(g, h) = (s_1 \cdots s_m \cdot t_1^{2i}, s'_1 \cdots s'_n \cdot t_1^{2k}) \\ = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (s_i, s'_j) = t_1^{2\alpha}.$$

THEOREM 2.1. *Let $M_n = \langle t_1, \dots, t_n \rangle$ be the Miller group on n generators. Then there exist nonabelian groups of order 8, G_1, \dots, G_k (where $k = k(n)$) such that if n is even then*

$$M_n \cong (G_1 \times \cdots \times G_k)_A,$$

whereas if $n \equiv 1(4)$

$$M_n \cong (G_1 \times \cdots \times G_k \times C_4)_A,$$

and if $n \equiv 3(4)$

$$M_n \cong (G_1 \times \cdots \times G_k)_A \times C_2,$$

where $(\quad)_A$ denotes the amalgamation of the square-generated subgroups of the factors and C_r is the cyclic group of order r .

Proof: Define a new set of generators as follows:

$$w_i = t_1 \cdots t_i \quad (i \geq 1) \text{ and, for } i \geq 3, \quad u_i = t_i t_{i+1} \text{ and } v_i = w_{i-2} t_i.$$

It is clear that

$$\left. \begin{aligned} M_{4k} &= \langle t_1, t_2, u_3, u_5, \dots, u_{4k-1}, v_4, v_6, \dots, v_{4k} \rangle \\ M_{4k+1} &= \langle M_{4k}, w_{4k+1} \rangle \end{aligned} \right\} k > 0$$

and

$$\left. \begin{aligned} M_{4k+2} &= \langle M_{4k}, u_{4k+1}, v_{4k+1} \rangle \\ M_{4k+3} &= \langle M_{4k+2}, w_{4k+3} \rangle \end{aligned} \right\} k \geq 0, \text{ where we set } M_0 = \langle 1 \rangle \\ \text{and } u_1 = t_1, v_1 = t_2.$$

Now let $S_1 = \langle t_1, t_2 \rangle, S_3 = \langle u_3, v_4 \rangle, \dots, S_{2k+1} = \langle u_{2k+1}, v_{2k} \rangle$. From Lemma 2.1. it follows that S_i is non-abelian of order 8 and that

$$\begin{aligned} M_{4k} &\cong (S_1^* \times \cdots \times S_{4k-1}^*)_A, & k > 0 \\ M_{4k+1} &\cong (S_1^* \times \cdots \times S_{4k-1}^* \times C_4)_A, & k > 0 \\ M_{4k+2} &\cong (S_1^* \times \cdots \times S_{4k+1}^*)_A \\ M_{4k+3} &\cong (S_1^* \times \cdots \times S_{4k+1}^*)_A \times C_2 \end{aligned}$$

where the S_i^* are disjoint copies of the $S_i, (\quad)_A$ denotes the amalgamation of the square-generated subgroups of the direct factors, and C_r is the cyclic subgroup of order r .

COROLLARY 2.1. $M_n \in \mathfrak{S}$ if and only if $n \equiv 1(4)$.

Proof. When $n \equiv 1(4)$ it follows from Theorem 2.1 that M_n has a central element of order 4; hence, in this case, $M_n \in \mathfrak{S}$. If $n \not\equiv 1(4)$ it follows from Theorem 2.1 that M_n is the direct product of either the identity group or C_2 with a factor group of a direct product of quaternion and dihedral groups. Since all of these groups are in \mathfrak{S} it follows that $M_n \in \mathfrak{S}$.

It is possible, of course, to prove Corollary 2.1 directly without appealing to Theorem 2.1. However, since Theorem 2.1 is itself of some interest it seems better to prove things in the order they are done here.

The author wishes to thank C. Hobby for many valuable suggestions.

BIBLIOGRAPHY

1. W. Burnside, *The Theory of groups of finite order*, (2nd ed.) 1911, Dover, New York, 1955.
2. M. Hall, *The theory of groups*, MacMillan, New York, 1959.
3. G. A. Miller, *Second note on the groups generated by operators transforming each other into their inverses*, Quart. J. Math. **44** (1913), 142-146.
4. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

Received October 25, 1967. This paper is part of the author's Ph. D. thesis. The author was supported by the Pre-Doctoral Traineeship Program of the National Aeronautics and Space Administration.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 28, No. 2

April, 1969

Richard Arens and Donald George Babbitt, <i>The geometry of relativistic n-particle interactions</i>	243
Kirby Alan Baker, <i>Hypotopological spaces and their embeddings in lattices with Birkhoff interval topology</i>	275
J. Lennart (John) Berggren, <i>Finite groups in which every element is conjugate to its inverse</i>	289
Beverly L. Brechner, <i>Homeomorphism groups of dendrons</i>	295
Robert Ray Colby and Edgar Andrews Rutter, <i>QF - 3 rings with zero singular ideal</i>	303
Stephen Daniel Comer, <i>Classes without the amalgamation property</i>	309
Stephen D. Fisher, <i>Bounded approximation by rational functions</i>	319
Robert Gaines, <i>Continuous dependence for two-point boundary value problems</i>	327
Bernard Russel Gelbaum, <i>Banach algebra bundles</i>	337
Moses Glasner and Richard Emanuel Katz, <i>Function-theoretic degeneracy criteria for Riemannian manifolds</i>	351
Fletcher Gross, <i>Fixed-point-free operator groups of order 8</i>	357
Sav Roman Harasymiv, <i>On approximation by dilations of distributions</i>	363
Cheong Seng Hoo, <i>Nilpotency class of a map and Stasheff's criterion</i>	375
Richard Emanuel Katz, <i>A note on extremal length and modulus</i>	381
H. L. Krall and I. M. Sheffer, <i>Difference equations for some orthogonal polynomials</i>	383
Yu-Lee Lee, <i>On the construction of lower radical properties</i>	393
Robert Phillips, <i>Liouville's theorem</i>	397
Yum-Tong Siu, <i>Analytic sheaf cohomology groups of dimension n of n-dimensional noncompact complex manifolds</i>	407
Michael Samuel Skaff, <i>Vector valued Orlicz spaces. II</i>	413
James DeWitt Stein, <i>Homomorphisms of B^*-algebras</i>	431
Mark Lawrence Teply, <i>Torsionfree injective modules</i>	441
Richard R. Tucker, <i>The δ^2-process and related topics. II</i>	455
David William Walkup and Roger Jean-Baptiste Robert Wets, <i>Lifting projections of convex polyhedra</i>	465
Thomas Paul Whaley, <i>Large sublattices of a lattice</i>	477