A NOTE ON EXTREMAL LENGTH AND MODULUS

RICHARD EMANUEL KATZ
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Richard Katz

The equality of extremal length and modulus is shown for general annuli in a Riemannian manifold.

In a recent paper, [2], Ow showed that the modulus and extremal length of an annulus in a Riemannian manifold are equal under the assumption that the set of critical points of the harmonic measure of the annulus has capacity zero. The purpose of this note is to show that this condition can be dispensed with. We refer the reader to [2] for definitions.

THEOREM. The extremal length \( \lambda \) and modulus \( \mu \) of an annulus \( (\Omega, \alpha, \beta) \) are equal.

Proof. The inequality \( \lambda \geq \mu \) was shown in [2].

To show the opposite inequality, let \( u \) be the harmonic measure of \( (\Omega, \alpha, \beta) \), and \( h \) be a function such that

\[
 h | \alpha = 0, \ h | \beta = 1, \ |\nabla h^2 - \nabla u^2| < \varepsilon ,
\]

and \( h \) has only a finite number of critical points in \( \Omega \). For the existence of such a function see Milnor [1]. Let \( \Gamma_0 \) be the set of integral curves of \( \nabla h \) which do not meet a critical point and \( P \) the set of admissible densities on \( \Omega \). It is immediate that \( |\nabla h| \in P \). Now for \( \gamma \in \Gamma_0, \rho \in P \)

\[
 \int_\Omega \rho^2 dV = \int_\alpha \left( \int_{\nabla h} \rho^2 dh \right) \ast dh \geq \int_\alpha \left( \int_{\nabla h} \rho dh \right)^2 dh
\]

and

\[
 \int_\alpha \left( \int_{\nabla h} \rho \right)^2 \ast dh = \int_\alpha \left( \int_{\gamma} \rho \right)^2 \ast dh \geq \inf_{\Gamma_0} \left( \int_{\gamma} \rho \right)^2 \ast dh .
\]

Since \( |\nabla h^2 - \nabla u^2| < \varepsilon \), it follows that

\[
 \left| \int_\alpha \ast dh - \mu^{-1} \right| < \varepsilon V(\Omega)
\]

and therefore,

\[
 \lambda \leq \sup_{\rho} \inf_{\Gamma_0} \frac{\left( \int_{\gamma} \rho \right)^2}{\int_\rho \rho^2 dV} \leq \mu + K\varepsilon
\]
for a suitable constant $K$. Since $\varepsilon$ was arbitrary, we have $\lambda \leq \mu$ which completes the proof.

REFERENCES


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Richard Arens and Donald George Babbitt, *The geometry of relativistic n-particle interactions* .......................................................... 243
Kirby Alan Baker, *Hypotopological spaces and their embeddings in lattices with Birkhoff interval topology* ........................................... 275
J. Lennart (John) Berggren, *Finite groups in which every element is conjugate to its inverse* .............................................................. 289
Beverly L. Brechner, *Homeomorphism groups of dendrons* .................. 295
Robert Ray Colby and Edgar Andrews Rutter, *QF − 3 rings with zero singular ideal* ................................................................. 303
Stephen Daniel Comer, *Classes without the amalgamation property* ........ 309
Stephen D. Fisher, *Bounded approximation by rational functions* ........... 319
Robert Gaines, *Continuous dependence for two-point boundary value problems* ................................................................................. 327
Bernard Russel Gelbaum, *Banach algebra bundles* ............................... 337
Moses Glasner and Richard Emanuel Katz, *Function-theoretic degeneracy criteria for Riemannian manifolds* ........................................ 351
Fletcher Gross, *Fixed-point-free operator groups of order 8* ..................... 357
Sav Roman Harasymiv, *On approximation by dilations of distributions* .... 363
Cheong Seng Hoo, *Nilpotency class of a map and Stasheff’s criterion* .... 375
Richard Emanuel Katz, *A note on extremal length and modulus* .......... 381
H. L. Krall and I. M. Sheffer, *Difference equations for some orthogonal polynomials* ........................................................................ 383
Yu-Lee Lee, *On the construction of lower radical properties* ................... 393
Robert Phillips, *Liouville’s theorem* ...................................................... 397
Yum-Tong Siu, *Analytic sheaf cohomology groups of dimension n of n-dimensional noncompact complex manifolds* .................................. 407
Michael Samuel Skaff, *Vector valued Orlicz spaces. II* .......................... 413
James DeWitt Stein, *Homomorphisms of B*-algebras* ........................... 431
Mark Lawrence Teply, *Torsionfree injective modules* ............................. 441
Richard R. Tucker, *The $\delta^2$-process and related topics. II* ............... 455
David William Walkup and Roger Jean-Baptiste Robert Wets, *Lifting projections of convex polyhedra* ............................................. 465
Thomas Paul Whaley, *Large sublattices of a lattice* .............................. 477