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**DIFFERENCE EQUATIONS FOR SOME ORTHOGONAL
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H. L. KRALL AND I. M. SHEFFER

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It is well-known that every orthogonal polynomial set $\{P_n(x)\}$ satisfies a 3-term recurrence relation of the form

$$(1.1) \quad P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x) \quad (n = 1, 2, \dots).$$

Some orthogonal sets (polynomials of Jacobi, Hermite and so on) are solutions of differential equations. It will be shown that there exist orthogonal polynomial sets that satisfy 3-term difference equations of the form

$$(1.2) \quad A(x)y(x + \alpha) + B(x)y(x - \alpha) + C(x)y(x) = \lambda y(x)$$

where A, B, C are polynomials of degree ≤ 2 and λ is a parameter.

Consider the difference equation

$$(1.3) \quad A(x)y(x + \alpha) + B(x)y(x + \beta) + C(x)y(x) = \lambda y(x)$$

where A, B, C are real polynomials, λ is a parameter, and $\alpha, \beta, 0$ are distinct and real. We examine two cases, according as A, B, C are of degree ≤ 1 :

(a) $A(x) = a_1 x + a_0, B(x) = b_1 x + b_0, C(x) = c_1 x,$
 or are of degree ≤ 2 :

(b) $A(x) = a_2 x^2 + a_1 x + a_0, B(x) = b_2 x^2 + b_1 x + b_0, C(x) = c_2 x^2 + c_1 x$
 $(a_2, b_2, c_2 \text{ not all zero}).$ We shall use the notation (1.3a), (1.3b) to denote equation (1.3) for the respective conditions (a), (b).

Equation (1.3) will be termed *admissible* if there exists a real sequence $\{\lambda_n\}$ ($n = 0, 1, \dots$) such that for $\lambda = \lambda_n$ there is a polynomial solution $y_n(x)$, unique to within a multiplicative constant, and $y_n(x)$ is of degree (exactly) n . It follows that admissibility implies that

$$(1.4) \quad \lambda_m \neq \lambda_n (m \neq n).$$

LEMMA 1.1. Equation (1.3a) is admissible if and only if

$$(1.5) \quad a_1 + b_1 + c_1 = 0, \quad \delta \equiv a_1 \alpha + b_1 \beta \neq 0.$$

And in this case we have

$$(1.6) \quad \lambda_n = (a_0 + b_0) + n(a_1 \alpha + b_1 \beta) \quad (n = 0, 1, \dots).$$

Proof. Let n be arbitrary. If we substitute the n th degree polynomial

$$(1.7) \quad y(x) = x^n + \sum_{j=0}^{n-1} p_j x^j$$

into (1.3a), a necessary and sufficient condition that $y(x)$ be a solution is that coefficients of x^{n+1}, x^n, \dots agree on both sides. The coefficient of x^{n+1} yields the first of (1.5), that of x^n gives $\lambda = \lambda_n$ as in (1.6); and those of x^{n-1}, \dots, x^0 give successive equations for p_{n-1}, \dots, p_0 . In these equations the coefficients of p_{n-1}, \dots, p_0 are respectively $\lambda_n - \lambda_{n-1}, \lambda_n - \lambda_{n-2}, \dots, \lambda_n - \lambda_0$, so there is one and only one choice of the p_j 's if and only if $\lambda_n \neq \lambda_j$ ($j \leq n-1$). This condition is equivalent to the second part of (1.5); and the lemma is established.

LEMMA 1.2. *Equation (1.3b) is admissible if and only if*

$$(1.8) \quad a_2 + b_2 + c_2 = 0, \quad a_1 + b_1 + c_1 = 0, \quad a_2\alpha + b_2\beta = 0;$$

$$(1.9) \quad 2(a_1\alpha + b_1\beta) + n(a_2\alpha^2 + b_2\beta^2) \neq 0 \quad (n = 0, 1, \dots).$$

And in this case λ_n is given by

$$(1.10) \quad \lambda_n = (a_0 + b_0) + n(a_1\alpha + b_1\beta) + n(n-1)(a_2\alpha^2 + b_2\beta^2)/2 \\ (n = 0, 1, \dots).$$

Proof. Substituting (1.7) into (1.3b) and equating like terms (as a necessary and sufficient condition for a solution) we find that the terms in x^{n+2}, x^{n+1} give (1.8), the x^n term gives $\lambda = \lambda_n$ as in (1.10), and p_{n-1}, \dots, p_0 again are uniquely determined if and only if $\lambda_n \neq \lambda_j$ ($j \leq n-1$). Now the condition $\lambda_m \neq \lambda_n$ ($m \neq n$) is seen to reduce to (1.9); so the lemma is proved.

In the proofs of Lemmas 1.1, 1.2 it was seen that if a polynomial $y(x)$ of degree n satisfies (1.3a or b) then the corresponding value of λ is λ_n as given by (1.6) or (1.10); so we have the

COROLLARY. *If (1.3a) or (1.3b) is admissible then for each $\lambda \neq \lambda_n$ ($n = 0, 1, \dots$) the only polynomial solution is $y(x) \equiv 0$.*

Let (1.3a) or (1.3b) be admissible. In both cases the solution for $n = 1$ is

$$(1.11) \quad y_1(x) = x + (a_0\alpha + b_0\beta)\delta^{-1}$$

where δ is given in (1.5). If we set

$$x + d = x^*, \quad z(x^*) = y(x^* - d)$$

with

$$d = (a_0\alpha + b_0\beta)\delta^{-1},$$

the equation in $z(x^*)$ will also be admissible and will have the form (1.3a) or (1.3b) after the constant term in $C(x^*)$ has been absorbed into the λ . Moreover, for $n = 1$ we have

$$z_1(x^*) = x^* .$$

An admissible equation (1.3a) or (1.3b) in which for $n = 1$ the solution contains no constant term will be called *canonical*. It is no restriction to limit ourselves to canonical equations.

From (1.11) we obtain

LEMMA 1.3. *The admissible equation (1.3a) or (1.3b) is canonical if and only if*

$$(1.12) \quad a_0\alpha + b_0\beta = 0 .$$

2. **Orthogonality for case (1.3a).** We consider the problem of determining those canonical equations (1.3a) [(1.3b) in § 3] whose polynomial solutions form an orthogonal set. For all polynomials $y(x)$ we have

$$(2.1) \quad y(x + u) = \sum_{k=0}^{\infty} y^{(k)}(x)u^k/k!$$

so (1.3a) is equivalent, with respect to polynomial solutions, to the differential equation of infinite order

$$(2.2) \quad xy'(x) + \sum_{k=2}^{\infty} H_k(x)y^{(k)}(x)/k! = \sigma y(x)$$

where

$$(2.3) \quad H_k(x) = r_k + s_kx = (a_0\alpha^k + b_0\beta^k)\delta^{-1} + (a_1\alpha^k + b_1\beta^k)\delta^{-1}x$$

($k = 1, 2, \dots$) with $\sigma = \{\lambda - (a_0 + b_0)\delta^{-1}$. Using (1.6) we find that the sequence $\{\sigma_n\}$ for which there are polynomial solutions is given by $\sigma_n = n$.

Equation (2.2) is identical with equation (3.1) of [1]. In Remark (i) ([1], p. 151) it is shown that if $r_2 = 0$ the polynomial solutions do not form an orthogonal set. We therefore assume $r_2 \neq 0$. In this case, Theorem 3.1 ([1], p. 151) states that the solutions of (our present) equation (2.2), hence of cononical equation (1.3a), form a weak orthogonal set if and only if

$$(2.4) \quad \begin{aligned} r_{2p+1} &= 0, & s_{2p+1} &= s_3^p, \\ r_{2p+2} &= r_2s_3^p, & s_{2p+2} &= s_2s_3^p \quad (p = 0, 1, \dots) . \end{aligned}$$

Moreover the weak orthogonal set is an orthogonal set when and only when one of the following two relations holds:

$$(2.5_1) \quad s_2^2 - s_3 = 0 ;$$

$$(2.5_2) \quad s_2^2 - s_3 \neq 0 \quad \text{and} \quad 2r_2(s_2^2 - s_3)^{-1} \neq 0, 1, 2, \dots .$$

The condition $r_{2p+1} = 0$ is

$$(2.6) \quad a_0\alpha^{2p+1} + b_0\beta^{2p+1} = 0 \quad (p = 0, 1, \dots) .$$

If $a_0 = 0$ or $b_0 = 0$ then both are zero since $\alpha\beta \neq 0$. But then $r_2 = 0$, contrary to assumption. So $a_0b_0 \neq 0$. Taking $p = 0, 1$ in (2.6) we then get $\beta^2 = \alpha^2$. Since α, β are distinct, then $\beta = -\alpha$; and again from (2.6) with $p = 0$: $a_0 = b_0$. Thus, if $r_2 \neq 0$ then $r_{2p+1} = 0$ ($p = 0, 1, \dots$) if and only if

$$(2.7) \quad \beta = -\alpha, a_0 = b_0 \neq 0 .$$

With (2.7) holding then

$$\delta = \alpha(a_1 - b_1) \neq 0 ,$$

so

$$(2.8) \quad \begin{aligned} r_{2p+1} &= 0, s_{2p+1} = \alpha^{2p}, r_{2p+2} = 2a_0(a_1 - b_1)^{-1}\alpha^{2p+1}, \\ s_{2p+2} &= (a_1 + b_1)(a_1 - b_1)^{-1}\alpha^{2p+1} . \end{aligned}$$

Conditions (2.4) are seen to be satisfied. And (2.5₁), (2.5₂) become respectively:

$$(2.9_1) \quad a_1b_1 = 0 ;$$

$$(2.9_2) \quad a_1b_1 \neq 0, a_0(a_1 - b_1)(\alpha a_1 b_1)^{-1} \neq 0, 1, \dots .$$

To sum up:

THEOREM 2.1. *Let equation (1.3a) be canonical. Then its polynomial solutions form an orthogonal set if and only if (2.7) holds and one of (2.9₁), (2.9₂) holds.*

REMARKS. (i) If (1.3a) is canonical its polynomial solutions form an orthogonal set if and only if it is of the form

$$(2.10) \quad \begin{aligned} (a_1x + a_0)y(x + \alpha) + (b_1x + a_0)y(x - \alpha) \\ - (a_1 + b_1)xy(x) = \lambda y(x) , \end{aligned}$$

with $a_0 \neq 0, a_1 \neq b_1, \alpha \neq 0$, and either (2.9₁) or (2.9₂) holding.

(ii) In (2.10) make the variable changes $x = \alpha x^*, z(x^*) = y(\alpha x^*)$. There results a similar difference equation in $z(x^*)$, in which α is replaced by 1. This equation has an orthogonal set of solutions when (2.10) does. It may be termed a *standard* canonical equation. After

dividing by a_0 this equation has the form (dropping asterisks)

$$(2.11) \quad (c_1x + 1)z(x + 1) + (d_1x + 1)z(x - 1) - (c_1 + d_1)xz(x) = \mu z(x),$$

with $c_1 - d_1 \neq 0$ and either $c_1d_1 = 0$ or

$$c_1d_1 \neq 0, (c_1 - d_1)(c_1d_1)^{-1} \neq 0, 1, 2, \dots .$$

3. Orthogonality for case (1.3b). Let equation (1.3b) be canonical, so that (1.12) holds. Putting (2.1) into (1.3b) we get an infinite order differential equation with polynomial coefficients of degree ≤ 2 , which is equivalent to (1.3b) at least for polynomial solutions:

$$(3.1) \quad xy'(x) + \sum_{k=2}^{\infty} T_k(x)y^{(k)}(x)/k! = \sigma y(x),$$

where

$$(3.2) \quad T_k(x) = r_k + s_kx + t_kx^2 = (a_0\alpha^k + b_0\beta^k)\delta^{-1} + (a_1\alpha^k + b_1\beta^k)\delta^{-1}x + (a_2\alpha^k + b_2\beta^k)\delta^{-1}x^2 \quad (k = 2, 3, \dots)$$

and $\sigma = \{\lambda - (a_0 + b_0)\}\delta^{-1}$ and δ is given by (1.5). From (1.10) we see that $\{\sigma_n\}$ is given by

$$\sigma_n = n + n(n - 1)t_2/2 .$$

Equations of the form (3.1), that is, with $\max_k \{\text{degree } T_k(x)\} = 2$ were considered in [1], but the results obtained were not as complete as for the case where the coefficients are of degree ≤ 1 . We must therefore proceed differently. We first show that if canonical equation (1.3b), hence also (3.1), has an orthogonal set of solutions then $\beta = -\alpha$.

For suppose not. Then $|\alpha| \neq |\beta|$, since α, β are distinct. We may assume that $|\alpha| > |\beta|$. By Theorem 2.2 ([1], p. 148) there is a sequence of constants $\{\alpha_n\}$ (the moments of the weight function corresponding to the orthogonal set), with $\alpha_0 \neq 0$, that satisfies the system of equations

$$(3.3) \quad d_{p+k}^p = 0, D_{p+k}^p = 0 \quad (p, k = 0, 1, \dots)$$

where (in our present case, as seen in [1], p. 153)

$$(3.4) \quad d_{p+k}^p = \sum_{i=k}^{2k+2} \alpha_i \left[\binom{k}{i-k} r_{2p+2k+1-i} + \binom{k}{i-k-1} s_{2p+2k+2-i} + \binom{k}{i-k-2} t_{2p+2k+3-i} \right],$$

$$\begin{aligned}
 D_{p+k}^p &= \sum_{i=k}^{2k+3} \alpha_i \left[\frac{i+1}{k+1} \binom{k+1}{i-k} r_{2p+2k+2-i} \right. \\
 (3.5) \quad &+ \frac{i}{k+1} \binom{k+1}{i-k-1} s_{2p+2k+3-i} \\
 &\left. + \frac{i-1}{k+1} \binom{k+1}{i-k-2} t_{2p+2k+4-i} \right].
 \end{aligned}$$

Here the convention is made that $\binom{m}{q} = 0$ for $q < 0$, and $r_j = s_j = t_j = 0$ for $j \leq 0$ and $r_1 = t_1 = 0, s_1 = 1$.

Putting the values of r_k, s_k, t_k from (3.2) into (3.3) we get

$$\begin{cases} \alpha^{2p+2k+1} U_k + \beta^{2p+2k+1} V_k = 0 \\ \alpha^{2p+2k+2} W_k + \beta^{2p+2k+2} X_k = 0 \end{cases} \quad (p, k = 0, 1, \dots)$$

where

$$\begin{aligned}
 U_k &= \sum_{i=k}^{2k+2} \alpha_i \left[\binom{k}{i-k} a_0 \alpha^{-i} + \binom{k}{i-k-1} a_1 \alpha^{-i+1} \right. \\
 &\quad \left. + \binom{k}{i-k-2} a_2 \alpha^{-i+2} \right], \\
 (3.7) \quad W_k &= \sum_{i=k}^{2k+3} \alpha_i \left[\frac{i+1}{k+1} \binom{k+1}{i-k} a_0 \alpha^{-i} + \frac{i}{k+1} \binom{k+1}{i-k-1} a_1 \alpha^{-i+1} \right. \\
 &\quad \left. + \frac{i-1}{k+1} \binom{k+1}{i-k-2} a_2 \alpha^{-i+2} \right],
 \end{aligned}$$

and V_k, X_k are obtained from U_k, W_k by replacing

$$a_0, a_1, a_2, \alpha \quad \text{by} \quad b_0, b_1, b_2, \beta.$$

Let k be arbitrary but fixed. If we divide (3.6) by $\alpha^{2p+2k+1}, \alpha^{2p+2k+2}$ respectively and let $p \rightarrow \infty$, then since $|\beta/\alpha| < 1$ we get

$$(3.8) \quad U_k = 0, W_k = 0 \quad (k = 0, 1, \dots).$$

And from (3.6) we then have

$$(3.9) \quad V_k = 0, X_k = 0 \quad (k = 0, 1, \dots).$$

For $k = 0$, (3.8), (3.9) reduce to

$$\begin{aligned}
 (3.10) \quad &\alpha_0 a_0 + \alpha_1 a_1 + \alpha_2 a_2 = 0, \alpha_1 a_0 + \alpha_2 a_1 + \alpha_3 a_2 = 0, \\
 &\alpha_0 b_0 + \alpha_1 b_1 + \alpha_2 b_2 = 0, \alpha_1 b_0 + \alpha_2 b_1 + \alpha_3 b_2 = 0.
 \end{aligned}$$

Now from (3.3) with $p = k = 0$ we have

$$\alpha_0 r_1 + \alpha_1 s_1 + \alpha_2 t_1 = 0.$$

But $r_1 = t_1 = 0, s_1 = 1$; hence

$$\alpha_1 = 0 .$$

So (3.10) becomes

$$(3.11) \quad \begin{aligned} \alpha_0 a_0 + \alpha_2 a_2 = 0, \alpha_2 a_1 + \alpha_3 a_2 = 0 , \\ \alpha_0 b_0 + \alpha_2 b_2 = 0, \alpha_2 b_1 + \alpha_3 b_2 = 0 . \end{aligned}$$

Now $a_2 b_2 \neq 0$. For if a_2 or b_2 is zero then from $a_2 \alpha + b_2 \beta = 0$ (in (1.8)) and $\alpha \beta \neq 0$ we get $a_2 = b_2 = 0$. Hence (again from (1.8)) $c_2 = 0$; so all coefficients in (1.3b) are of degree < 2 , contrary to assumption. Again, $a_0 b_0 \neq 0$. For if a_0 or b_0 is zero then (3.11) implies that $\alpha_2 = 0$. Since we already have $\alpha_1 = 0$, then $\Delta_1 = \begin{vmatrix} \alpha_0 \alpha_1 \\ \alpha_1 \alpha_2 \end{vmatrix} = 0$. But for the moments $\{\alpha_n\}$ corresponding to an orthogonal set it is known [2] that

$$\Delta_n \equiv \begin{vmatrix} \alpha_0 \alpha_1 \cdots \alpha_n \\ \alpha_1 \alpha_2 \cdots \alpha_{n+1} \\ \dots \dots \dots \\ \alpha_n \alpha_{n+1} \cdots \alpha_{2n} \end{vmatrix} \neq 0 \quad (n = 0, 1, \dots) ;$$

so we have a contradiction. Thus,

$$(3.12) \quad a_2 b_2 \neq 0, a_0 b_0 \neq 0, \alpha_2 \neq 0 .$$

The right hand equations in (3.11) give us

$$-b_1 a_2 + a_1 b_2 = 0 .$$

This with

$$\alpha a_2 + \beta b_2 = 0$$

from (1.8) implies

$$\alpha a_1 + \beta b_1 = 0 ,$$

contrary to (1.9) for $n = 0$. So the assumption $\beta \neq -\alpha$ leads to a contradiction, and we have

$$(3.13) \quad \beta = -\alpha .$$

Then from (1.12):

$$(3.14) \quad a_0 = b_0 .$$

In (3.2) we now have

$$(3.15) \quad \begin{cases} r_{2p} = 2a_0 \delta^{-1} \alpha^{2p}, s_{2p} = (a_1 + b_1) \delta^{-1} \alpha^{2p}, t_{2p} = (a_2 + b_2) \delta^{-1} \alpha^{2p} \\ r_{2p+1} = 0, s_{2p+1} = (a_1 - b_1) \delta^{-1} \alpha^{2p+1}, t_{2p+1} = (a_2 - b_2) \delta^{-1} \alpha^{2p+1} \end{cases}$$

($p = 1, 2, \dots$), with $r_1 = t_1 = 0, s_1 = 1$. (3.12) and (3.15) show that

$$r_2 \neq 0.$$

Let

$$(3.16) \quad u_p = s_{2p+1}, v_p = t_{2p+1}, w_p = t_{2p+2}.$$

From $a_2\alpha + b_2\beta = 0, \beta = -\alpha \neq 0$ we get

$$(3.17) \quad a_2 = b_2.$$

It is then readily seen that

$$(3.18) \quad v_p = 0, w_p - t_2 u_p = 0 \quad (p = 0, 1, 2, \dots)$$

Choose r_2, s_2, t_2, s_3 to satisfy the conditions

$$(3.19) \quad r_2 \neq 0, 2 + kt_2 \neq 0 (k = 0, 1, \dots), \Delta_2 \neq 0,$$

where $\alpha_1 = 0, \alpha_2, \alpha_3, \alpha_4$ are obtained from the equations

$$(3.20) \quad D_0^0 = 0, d_1^0 = 0, D_1^0 = 0.$$

(3.18)–(3.20) make Theorem 4.2 ([1], p. 158) applicable, so that the solutions of (1.3b) form a weak orthogonal set if and only if

$$(3.21) \quad \begin{cases} s_{2p+1} = s_3^p, t_{2p+1} = 0, r_{2p+1} = 0 \\ s_{2p+2} = s_2 s_3^p, t_{2p+2} = t_2 s_3^p, r_{2p+2} = r_2 s_3^p \end{cases} \quad (p = 0, 1, \dots).$$

Now these conditions do hold in view of (3.14).

The first two conditions of (3.19) become

$$(3.22) \quad a_0 \neq 0; (a_1 - b_1) + k\alpha a_2 \neq 0 \quad (k = 0, 1, 2, \dots).$$

Finally, for weak orthogonality to imply orthogonality it is necessary and sufficient ([1], pp. 161–162) that $t_2 \notin S(r_2, s_2, s_3)$ where $S(r_2, s_2, s_3)$ is the set of all real values of t_2 for which $\pi_n(r_2, s_2, s_3, t_2) = 0$ for some $n > 1$. The expression for π_n is lengthy, and we do not reproduce it here. We merely observe that for given r_2, s_2, s_3 the set $S(r_2, s_2, s_3)$ is at most denumerable.

To sum up:

THEOREM 3.1. *Let the admissible equation (1.3b) be canonical. Its solutions form an orthogonal polynomial set if and only if:*

- (i) (3.12), (3.13), (3.14), (3.17), (3.19) hold.
- (ii) $t_2 \notin S(r_2, s_2, s_3)$.

REMARKS. (a) If the canonical equation (1.3b) has an orthogonal polynomial set of solutions then it has the form

$$(3.23) \quad (a_2x^2 + a_1x + a_0)y(x + \alpha) + (a_2x^2 + b_1x + a_0)y(x - \alpha) - [2a_2x^2 + (a_1 + b_1)x]y(x) = \lambda y(x) ,$$

with

$$(3.24) \quad a_0a_2(a_1 - b_1)\alpha \neq 0; (a_1 - b_1) + k\alpha a_2 \neq 0 \quad (k = 0, 1, \dots) .$$

(b) As in § 2 the transformation $x = \alpha x^*, z(x^*) = y(\alpha x^*)$ carries (3.24) into a similar equation with α replaced by 1.

4. **Two examples.** If an orthogonal polynomial set $\{P_n(x)\}$ satisfies (2.10) with $\lambda = \lambda_n$ for $y = P_n(x)$ then from (1.6) we have

$$(4.1) \quad \lambda_n = 2a_0 + n\alpha(a_1 - b_1) \quad (n = 0, 1, \dots) .$$

Let $\{P_n(x)\}, \{Q_n(x)\}$ be polynomial sets defined by the respective generating functions

$$(4.2) \quad e^{ct}(1 - t)^{x+c} = \sum_{n=0}^{\infty} P_n(x)t^n \quad (c \neq 0) ,$$

$$(4.3) \quad (1 - t)^{x-bd} \cdot (1 - bt)^{-x+d} = \sum_{n=0}^{\infty} Q_n(x)t^n \quad (b \neq 0, 1) .$$

We shall show that these sets are orthogonal and satisfy an equation of the form (2.10).

Denote the left side of (2.10) by $L[y]$. If $G(x, t)$ is the generating function in (4.2) then

$$(4.4) \quad L[G] = G\{(a_1x + a_0)(1 - t)^\alpha + (b_1x + a_0)(1 - t)^{-\alpha} - (a_1 + b_1)x\} .$$

Also,

$$(4.5) \quad \sum_{n=0}^{\infty} \lambda_n P_n(x)t^n = 2a_0G + \alpha(a_1 - b_1)t \partial G/\partial t = G\{2a_0 + \alpha(a_1 - b_1)t[c - (x + c)(1 - t)^{-1}]\} .$$

$\{P_n(x)\}$ will satisfy (2.10) if (4.4) and (4.5) are identical. It is a straightforward computation to show that they are identical if

$$(4.6) \quad \alpha = 1; a_1 = 0; b_1 = a_0/c .$$

Hence $\{P_n(x)\}$ is an orthogonal set which satisfies the equation

$$(4.7) \quad P_n(x + 1) + (x + c)P_n(x - 1) - xP_n(x) = (2c - n)P_n(x) .$$

In the same way it is found that $\{Q_n(x)\}$ is an orthogonal set that is a solution of (2.10) for

$$(4.8) \quad \alpha = 1; a_1 = bb_1; a_n = -bdb_1 .$$

The equation reduces to

$$(4.9) \quad \begin{aligned} b(x-d)Q_n(x+1) + (x-bd)Q_n(x-1) - (b+1)xQ_n(x) \\ = \{-2bd + n(b-1)\}Q_n(x). \end{aligned}$$

In the case of (4.9) the condition (2.9₂) is to hold. It reduces to

$$(4.10) \quad -d(b-1) \neq 0, 1, 2, \dots .$$

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