

Pacific Journal of Mathematics

ON THE CONSTRUCTION OF LOWER RADICAL PROPERTIES

YU-LEE LEE

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**The purpose of this paper is to give a simple construction
 of the lower radical properties for an arbitrary class of rings.**

Let \mathcal{S} be a class of rings. We shall say that the ring R is an \mathcal{S} -ring if R is in \mathcal{S} . An ideal J of R will be called an \mathcal{S} -ideal if J is an \mathcal{S} -ring. A ring which does not contain any nonzero \mathcal{S} -ideals will be called \mathcal{S} -semisimple. We shall call \mathcal{S} a radical property if the following three conditions hold:

- (A) homomorphic image of an \mathcal{S} -ring is an \mathcal{S} -ring,
- (B) every ring R contains a largest \mathcal{S} -ideal S ,
- (C) the quotient ring R/S is \mathcal{S} -semi-simple.

The largest \mathcal{S} -ideal S of a ring R is called the \mathcal{S} -radical of R .

Given a class of rings \mathcal{A} , Kurosh has constructed a lower radical property $\mathcal{L}(\mathcal{A})$ determined by \mathcal{A} , [1], [2], i.e., $\mathcal{L}(\mathcal{A})$ is a radical property, $\mathcal{A} \subseteq \mathcal{L}(\mathcal{A})$, and if \mathcal{T} is any radical property and $\mathcal{A} \subseteq \mathcal{T}$ then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{T}$.

In this paper we are going to give a simpler construction.

The construction is similar to [3], where we take \mathcal{A} to be the class of all nilpotent rings. It is proven in [3] that this construction is exactly the lower radical property determined by the class of nilpotent rings. We want to extend this construction to any class of rings.

Let \mathcal{A} be a class of ring and let \mathcal{A}_0 be the class of all homomorphic images of rings in \mathcal{A} . For each ring R , let $D_1(R)$ be the set of all ideals of R , and by induction, we define $D_{n+1}(R)$ to be the family of all rings which are ideals of some ring in $D_n(R)$ and set

$$D(R) = \cup \{D_n(R) : n = 1, 2, 3, \dots\} .$$

A ring R is called a $\mathcal{L}(\mathcal{A})$ -ring if $D(R/I)$ contains a nonzero ring which is isomorphic to a ring in \mathcal{A}_0 for each ideal I of R and $I \neq R$. The following facts are clear.

LEMMA 1. $\mathcal{A} \subseteq \mathcal{A}_0 \subseteq \mathcal{L}(\mathcal{A})$.

LEMMA 2. If I is an ideal of R then $D(I) \subseteq D(R)$.

LEMMA 3. Every isomorphic image of an $\mathcal{L}(\mathcal{A})$ -ring is an $\mathcal{L}(\mathcal{A})$ -ring.

LEMMA 4. *If A is isomorphic to B and $D(A)$ contains a ring which is isomorphic to a nonzero ring in \mathcal{A}_0 then so does $D(B)$.*

LEMMA 5. *If $\mathcal{A} \subseteq \mathcal{B}$ then $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.*

Also we need the following fact [1].

LEMMA 6. *A class of rings \mathcal{S} is a radical property if and only if*

(A) *A homomorphic image of an \mathcal{S} -ring is an \mathcal{S} -ring.*

(D) *If every nonzero homomorphic image of a ring R contains a nonzero \mathcal{S} -ideal, then R is an \mathcal{S} -ring.*

LEMMA 7. *If \mathcal{S} is a radical property, then for any ring R and any ideal I of R , the \mathcal{S} -radical of I is an ideal of R .*

THEOREM 1. *If \mathcal{A} is a class of rings, then $\mathcal{L}(\mathcal{A})$, constructed above, is a radical property.*

Proof. If R is in $\mathcal{L}(\mathcal{A})$ and I is any ideal of R . Consider the quotient ring R/I and any proper ideal J/I of R/I , $R/I/J/I \cong R/J$.

By definition, $D(R/J)$ contains a ring which is isomorphic to a nonzero ring in \mathcal{A}_0 and therefore so does $D(R/I/J/I)$, and hence R/I is in $\mathcal{L}(\mathcal{A})$. Every homomorphic image of R is isomorphic with R/I for some I . Hence, by Lemma 3, (A) follows.

Suppose that every nonzero homomorphic image of R contains a nonzero $\mathcal{L}(\mathcal{A})$ -ideal and let I be any ideal of R and $I \neq R$. Then R/I contains a nonzero \mathcal{L} -ideal J/I . Now $D(J/I) \subseteq D(R/I)$, hence $D(R/I)$ contains a ring which is isomorphic to a nonzero ring in \mathcal{A}_0 . By definition of $\mathcal{L}(\mathcal{A})$, R is in $\mathcal{L}(\mathcal{A})$. This proves (D). By Lemma 6, $\mathcal{L}(\mathcal{A})$ is a radical property.

THEOREM 2. *If \mathcal{T} is a radical property then $\mathcal{L}(\mathcal{T}) = \mathcal{T}$.*

Proof. By Lemma 1, $\mathcal{T} \subseteq \mathcal{L}(\mathcal{T})$.

If there is a ring R in $\mathcal{L}(\mathcal{T})$ but not in \mathcal{T} , let I be \mathcal{T} -radical of R . Then R/I is a nonzero ring in $\mathcal{L}(\mathcal{T})$ and is \mathcal{T} -semi-simple. Without loss of generality we may assume R is in $\mathcal{L}(\mathcal{T})$ but is \mathcal{T} -semi-simple. By definition $D(R)$ contains a ring $J \neq 0$ such that $J \in \mathcal{T}$. But if K is a nonzero ideal of R , i.e., $K \in D_1(R)$, then, by Lemma 7, the \mathcal{T} -radical of K is also an ideal of R . But R is \mathcal{T} -semi-simple. Hence K is also \mathcal{T} -semi-simple. By induction it is easy to see every ring in $D(R)$ is \mathcal{T} -semi-simple. This is a contradiction. Hence $\mathcal{T} = \mathcal{L}(\mathcal{T})$.

THEOREM 3. *If \mathcal{A} is a class of rings then $\mathcal{L}(\mathcal{A})$ is the lower radical property determined by \mathcal{A} .*

Proof. Let \mathcal{S} be any radical property such that $\mathcal{A} \subseteq \mathcal{S}$. Then by Theorem 2 and Lemma 5 $\mathcal{S} = \mathcal{L}(\mathcal{S}) \supseteq \mathcal{L}(\mathcal{A})$.

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