

# Pacific Journal of Mathematics

**ON THE CONSTRUCTION OF LOWER RADICAL PROPERTIES**

YU-LEE LEE

## ON THE CONSTRUCTION OF LOWER RADICAL PROPERTIES

YU-LEE LEE

**The purpose of this paper is to give a simple construction  
 of the lower radical properties for an arbitrary class of rings.**

Let  $\mathcal{S}$  be a class of rings. We shall say that the ring  $R$  is an  $\mathcal{S}$ -ring if  $R$  is in  $\mathcal{S}$ . An ideal  $J$  of  $R$  will be called an  $\mathcal{S}$ -ideal if  $J$  is an  $\mathcal{S}$ -ring. A ring which does not contain any nonzero  $\mathcal{S}$ -ideals will be called  $\mathcal{S}$ -semisimple. We shall call  $\mathcal{S}$  a radical property if the following three conditions hold:

- (A) homomorphic image of an  $\mathcal{S}$ -ring is an  $\mathcal{S}$ -ring,
- (B) every ring  $R$  contains a largest  $\mathcal{S}$ -ideal  $S$ ,
- (C) the quotient ring  $R/S$  is  $\mathcal{S}$ -semi-simple.

The largest  $\mathcal{S}$ -ideal  $S$  of a ring  $R$  is called the  $\mathcal{S}$ -radical of  $R$ .

Given a class of rings  $\mathcal{A}$ , Kurosh has constructed a lower radical property  $\mathcal{L}(\mathcal{A})$  determined by  $\mathcal{A}$ , [1], [2], i.e.,  $\mathcal{L}(\mathcal{A})$  is a radical property,  $\mathcal{A} \subseteq \mathcal{L}(\mathcal{A})$ , and if  $\mathcal{F}$  is any radical property and  $\mathcal{A} \subseteq \mathcal{F}$  then  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{F}$ .

In this paper we are going to give a simpler construction.

The construction is similar to [3], where we take  $\mathcal{A}$  to be the class of all nilpotent rings. It is proven in [3] that this construction is exactly the lower radical property determined by the class of nilpotent rings. We want to extend this construction to any class of rings.

Let  $\mathcal{A}$  be a class of ring and let  $\mathcal{A}_0$  be the class of all homomorphic images of rings in  $\mathcal{A}$ . For each ring  $R$ , let  $D_1(R)$  be the set of all ideals of  $R$ , and by induction, we define  $D_{n+1}(R)$  to be the family of all rings which are ideals of some ring in  $D_n(R)$  and set

$$D(R) = \cup \{D_n(R): n = 1, 2, 3, \dots\}.$$

A ring  $R$  is called a  $\mathcal{L}(\mathcal{A})$ -ring if  $D(R/I)$  contains a nonzero ring which is isomorphic to a ring in  $\mathcal{A}_0$  for each ideal  $I$  of  $R$  and  $I \neq R$ . The following facts are clear.

LEMMA 1.  $\mathcal{A} \subseteq \mathcal{A}_0 \subseteq \mathcal{L}(\mathcal{A})$ .

LEMMA 2. If  $I$  is an ideal of  $R$  then  $D(I) \subseteq D(R)$ .

LEMMA 3. Every isomorphic image of an  $\mathcal{L}(\mathcal{A})$ -ring is an  $\mathcal{L}(\mathcal{A})$ -ring.

LEMMA 4. *If  $A$  is isomorphic to  $B$  and  $D(A)$  contains a ring which is isomorphic to a nonzero ring in  $\mathcal{A}_0$  then so does  $D(B)$ .*

LEMMA 5. *If  $\mathcal{A} \subseteq \mathcal{B}$  then  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ .*

Also we need the following fact [1].

LEMMA 6. *A class of rings  $\mathcal{S}$  is a radical property if and only if*

(A) *A homomorphic image of an  $\mathcal{S}$ -ring is an  $\mathcal{S}$ -ring.*

(D) *If every nonzero homomorphic image of a ring  $R$  contains a nonzero  $\mathcal{S}$ -ideal, then  $R$  is an  $\mathcal{S}$ -ring.*

LEMMA 7. *If  $\mathcal{S}$  is a radical property, then for any ring  $R$  and any ideal  $I$  of  $R$ , the  $\mathcal{S}$ -radical of  $I$  is an ideal of  $R$ .*

THEOREM 1. *If  $\mathcal{A}$  is a class of rings, then  $\mathcal{L}(\mathcal{A})$ , constructed above, is a radical property.*

*Proof.* If  $R$  is in  $\mathcal{L}(\mathcal{A})$  and  $I$  is any ideal of  $R$ . Consider the quotient ring  $R/I$  and any proper ideal  $J/I$  of  $R/I$ ,  $R/I/J/I \cong R/J$ .

By definition,  $D(R/J)$  contains a ring which is isomorphic to a nonzero ring in  $\mathcal{A}_0$  and therefore so does  $D(R/I/J/I)$ , and hence  $R/I$  is in  $\mathcal{L}(\mathcal{A})$ . Every homomorphic image of  $R$  is isomorphic with  $R/I$  for some  $I$ . Hence, by Lemma 3, (A) follows.

Suppose that every nonzero homomorphic image of  $R$  contains a nonzero  $\mathcal{L}(\mathcal{A})$ -ideal and let  $I$  be any ideal of  $R$  and  $I \neq R$ . Then  $R/I$  contains a nonzero  $\mathcal{L}$ -ideal  $J/I$ . Now  $D(J/I) \subseteq D(R/I)$ , hence  $D(R/I)$  contains a ring which is isomorphic to a nonzero ring in  $\mathcal{A}_0$ . By definition of  $\mathcal{L}(\mathcal{A})$ ,  $R$  is in  $\mathcal{L}(\mathcal{A})$ . This proves (D). By Lemma 6,  $\mathcal{L}(\mathcal{A})$  is a radical property.

THEOREM 2. *If  $\mathcal{T}$  is a radical property then  $\mathcal{L}(\mathcal{T}) = \mathcal{T}$ .*

*Proof.* By Lemma 1,  $\mathcal{T} \subseteq \mathcal{L}(\mathcal{T})$ .

If there is a ring  $R$  in  $\mathcal{L}(\mathcal{T})$  but not in  $\mathcal{T}$ , let  $I$  be  $\mathcal{T}$ -radical of  $R$ . Then  $R/I$  is a nonzero ring in  $\mathcal{L}(\mathcal{T})$  and is  $\mathcal{T}$ -semi-simple. Without loss of generality we may assume  $R$  is in  $\mathcal{L}(\mathcal{T})$  but is  $\mathcal{T}$ -semi-simple. By definition  $D(R)$  contains a ring  $J \neq 0$  such that  $J \in \mathcal{T}$ . But if  $K$  is a nonzero ideal of  $R$ , i.e.,  $K \in D_1(R)$ , then, by Lemma 7, the  $\mathcal{T}$ -radical of  $K$  is also an ideal of  $R$ . But  $R$  is  $\mathcal{T}$ -semi-simple. Hence  $K$  is also  $\mathcal{T}$ -semi-simple. By induction it is easy to see every ring in  $D(R)$  is  $\mathcal{T}$ -semi-simple. This is a contradiction. Hence  $\mathcal{T} = \mathcal{L}(\mathcal{T})$ .

**THEOREM 3.** *If  $\mathcal{A}$  is a class of rings then  $\mathcal{L}(\mathcal{A})$  is the lower radical property determined by  $\mathcal{A}$ .*

*Proof.* Let  $\mathcal{S}$  be any radical property such that  $\mathcal{A} \subseteq \mathcal{S}$ . Then by Theorem 2 and Lemma 5  $\mathcal{S} = \mathcal{L}(\mathcal{S}) \supseteq \mathcal{L}(\mathcal{A})$ .

#### BIBLIOGRAPHY

1. N. J. Divinsky, *Rings and radicals*, University of Toronto Press, 1965.
2. A. G. Kurash, *Radicals of rings and algebras*, Mat. Sbornik (1953).
3. Y. L. Lee, *A characterization of Baer lower radical property*, Kyungpook Math. J. **7** (1967).

Received January 19, 1968. Presented to the Society on January 26, 1967.

KANSAS STATE UNIVERSITY  
MANHATTAN, KANSAS



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN  
Stanford University  
Stanford, California

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. R. PHELPS  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Richard Arens and Donald George Babbitt, <i>The geometry of relativistic <math>n</math>-particle interactions</i> .....	243
Kirby Alan Baker, <i>Hypotopological spaces and their embeddings in lattices with Birkhoff interval topology</i> .....	275
J. Lennart (John) Berggren, <i>Finite groups in which every element is conjugate to its inverse</i> .....	289
Beverly L. Brechner, <i>Homeomorphism groups of dendrons</i> .....	295
Robert Ray Colby and Edgar Andrews Rutter, <i>QF – 3 rings with zero singular ideal</i> .....	303
Stephen Daniel Comer, <i>Classes without the amalgamation property</i> .....	309
Stephen D. Fisher, <i>Bounded approximation by rational functions</i> .....	319
Robert Gaines, <i>Continuous dependence for two-point boundary value problems</i> .....	327
Bernard Russel Gelbaum, <i>Banach algebra bundles</i> .....	337
Moses Glasner and Richard Emanuel Katz, <i>Function-theoretic degeneracy criteria for Riemannian manifolds</i> .....	351
Fletcher Gross, <i>Fixed-point-free operator groups of order 8</i> .....	357
Sav Roman Harasymiv, <i>On approximation by dilations of distributions</i> .....	363
Cheong Seng Hoo, <i>Nilpotency class of a map and Stasheff's criterion</i> .....	375
Richard Emanuel Katz, <i>A note on extremal length and modulus</i> .....	381
H. L. Krall and I. M. Sheffer, <i>Difference equations for some orthogonal polynomials</i> .....	383
Yu-Lee Lee, <i>On the construction of lower radical properties</i> .....	393
Robert Phillips, <i>Liouville's theorem</i> .....	397
Yum-Tong Siu, <i>Analytic sheaf cohomology groups of dimension <math>n</math> of <math>n</math>-dimensional noncompact complex manifolds</i> .....	407
Michael Samuel Skaff, <i>Vector valued Orlicz spaces. II</i> .....	413
James DeWitt Stein, <i>Homomorphisms of <math>B^*</math>-algebras</i> .....	431
Mark Lawrence Teply, <i>Torsionfree injective modules</i> .....	441
Richard R. Tucker, <i>The <math>\delta^2</math>-process and related topics. II</i> .....	455
David William Walkup and Roger Jean-Baptiste Robert Wets, <i>Lifting projections of convex polyhedra</i> .....	465
Thomas Paul Whaley, <i>Large sublattices of a lattice</i> .....	477