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HOMOMORPHISMS OF B^* -ALGEBRAS

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This paper is divided into two sections. The first deals with Banach algebra homomorphisms of a von Neumann algebra \mathfrak{A} , and extends the Bade-Curtis theory for commutative B^* -algebras to von Neumann algebras, as well as characterizing the separating ideal in the closure of the range of the homomorphism. The second section concerns homomorphisms of B^* -algebras; the chief result being the existence of an ideal \mathcal{I} with cofinite closure such that the restriction of the homomorphism to any closed, two-sided ideal contained in \mathcal{I} is continuous.

1. Homomorphisms of von Neumann algebras. Let \mathfrak{A} be a von Neumann algebra, and let $\nu: \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism. The reduction theory enables us to write

$$\mathfrak{A} = \sum_{i=1}^{\infty} \oplus (C(X_i) \otimes B(\mathcal{H}_i)) \oplus \mathfrak{A}_1,$$

where \mathfrak{A}_1 is the direct sum of the type II and type III parts, X_i is a hyperstonian compact Hausdorff space, and \mathcal{H}_i is Hilbert space of dimension i (∞ is an allowed index of i , \mathcal{H}_{∞} is separable Hilbert space). It was shown in [6] that there is an integer N such that

$$\nu \Big| \sum_{i=N+1}^{\infty} \oplus (C(X_i) \otimes B(\mathcal{H}_i)) \oplus \mathfrak{A}_1$$

is continuous.

Some definitions are in order.

$$S(\nu, \mathfrak{B}) = \{z \in \mathfrak{B} \mid \exists \{x_n\} \subset \mathfrak{A} \ni x_n \rightarrow 0, \quad \nu(x_n) \rightarrow z\};$$

$S(\nu, \mathfrak{B})$ is a closed, 2-sided ideal in \mathfrak{B} ([2]). If $f \in C(X_i)$, $T \in B(\mathcal{H}_i)$, then $\langle f \otimes T \rangle$ will denote $(x, y) \in \mathfrak{A}$, where $y = 0 \in \mathfrak{A}_1$ and

$$x \in \sum_{k=1}^{\infty} \oplus (C(X_k) \otimes B(\mathcal{H}_k))$$

has $f \otimes T$ in the i^{th} component and zero in all other components. Let $\varphi_i: C(X_i) \rightarrow \mathfrak{B}$ be defined by $\varphi_i(f) = \nu(\langle f \otimes I_i \rangle)$, where I_i is the identity of $B(\mathcal{H}_i)$, and let F_i be the Bade-Curtis [1] singularity set associated with φ_i . Let $M(F_i) = \{f \in C(X_i) \mid f(F_i) = 0\}$, let $T(F_i) = \{f \in C(X_i) \mid f \text{ vanishes on a neighborhood of } F_i\}$, and let $R(F_i) = \{f \in C(X_i) \mid f \text{ is constant in a neighborhood of each point of } F_i\}$. It was shown in [6] that ν is continuous on

$$\sum_{i=1}^N \oplus (R(F_i) \otimes B(\mathcal{H}_i)) \oplus \sum_{i=N+1}^{\infty} \oplus (C(X_i) \otimes B(\mathcal{H}_i)) \oplus \mathfrak{A}_1 ,$$

and that this sub-algebra, denoted by \mathfrak{A}_0 , is dense in \mathfrak{A} . Let μ be the unique continuous extension of $\nu|_{\mathfrak{A}_0}$ to \mathfrak{A} and let $\lambda = \nu - \mu$. In this section the Bade-Curtis results ([1], Theorems 4.3 and 4.5) will be extended to \mathfrak{A} , and a complete characterization of $S(\nu, \mathfrak{B})$ will be obtained.

THEOREM 1.1. (a) *The range of μ is closed in \mathfrak{B} and $\overline{\nu(\mathfrak{A})} = \mu(\mathfrak{A}) \oplus S(\nu, \mathfrak{B})$, the direct sum being topological.*

(b) $S(\nu, \mathfrak{B}) = \overline{\lambda(\mathfrak{A})}$.

(c) *Let*

$$M = \sum_{i=1}^N \oplus (M(F_i) \otimes B(\mathcal{H}_i)) \oplus \sum_{i=N+1}^{\infty} \oplus (C(X_i) \otimes B(\mathcal{H}_i)) \oplus \mathfrak{A}_1 .$$

Then $S(\nu, \mathfrak{B}) \cdot M = M \cdot S(\nu, \mathfrak{B}) = (0)$, and $\lambda|_M$ is a homomorphism.

Proof. $\mu(\mathfrak{A})$ is closed by [2], Lemma 5.3. We first show $\lambda(\mathfrak{A}) \subseteq S(\nu, \mathfrak{B})$. If $x \in \mathfrak{A}$, choose a sequence $\{x_n\}$ from the dense sub-algebra such that $\lim_{n \rightarrow \infty} x_n = x$. Since μ is continuous,

$$\mu(x) = \lim_{n \rightarrow \infty} \mu(x_n) = \lim_{n \rightarrow \infty} \nu(x_n) ,$$

and since $\lim_{n \rightarrow \infty} (x_n - x) = 0$,

$$\mu(x) - \nu(x) = \lim_{n \rightarrow \infty} (\nu(x_n) - \nu(x)) = \lim_{n \rightarrow \infty} \nu(x_n - x) = s \in S(\nu, \mathfrak{B}) .$$

But $\nu(x) = \mu(x) + \lambda(x)$ and $\nu(x) = \mu(x) - s$, so $\lambda(x) = -s \in S(\nu, \mathfrak{B})$.

If $s \in S(\nu, \mathfrak{B})$, there is a sequence $\{x_n\}$ in \mathfrak{A} such that

$$\lim_{n \rightarrow \infty} x_n = 0, \quad \lim_{n \rightarrow \infty} \nu(x_n) = s .$$

Now $\lim_{n \rightarrow \infty} \mu(x_n) = 0$, and $s = \lim_{n \rightarrow \infty} (\mu(x_n) + \lambda(x_n))$, so

$$\|s - \lambda(x_n)\| \leq \|s - (\lambda(x_n) + \mu(x_n))\| + \|\mu(x_n)\| \rightarrow 0 ,$$

and so $S(\nu, \mathfrak{B}) = \overline{\lambda(\mathfrak{A})}$.

Let $U = \nu^{-1}(S(\nu, \mathfrak{B}))$. We now show $\mu(\mathfrak{A}) \cap S(\nu, \mathfrak{B}) = (0)$. If $\mu(x) \in S(\nu, \mathfrak{B})$, since $\nu(x) = \mu(x) + \lambda(x)$ and $\lambda(\mathfrak{A}) \subseteq S(\nu, \mathfrak{B})$, we see that $\nu(x) \in S(\nu, \mathfrak{B})$, and so $x \in U$. But by [6], Theorem II. 5, and [7], Proposition 2.1, $U = \overline{\text{Ker } \nu} = \text{Ker } (\mu)$, so $\mu(x) = 0$.

To complete the proof of (a) and (b), all we need show is that any $z \in \overline{\nu(\mathfrak{A})}$ can be written $z = \mu(x) + s$, where $x \in \mathfrak{A}$, $s \in S(\nu, \mathfrak{B})$. Let $\hat{\nu}: \mathfrak{A}/U \rightarrow \nu(\mathfrak{A})/S(\nu, \mathfrak{B})$ be defined by $\hat{\nu}(x + U) = \nu(x) + S(\nu, \mathfrak{B})$, by [2], Theorem 4.6, and [5], Theorem 4.9.2, this is a continuous

isomorphism of a B^* -algebra and thus has closed range. So $z + S(\nu, \mathfrak{B}) = \nu(x) + S(\nu, \mathfrak{B})$, and so $\exists s \in S(\nu, \mathfrak{B})$ such that $z = \nu(x) + s = \mu(x) + (\lambda(x) + s)$. But $\lambda(x) + s \in S(\nu, \mathfrak{B})$.

Define T by substituting $T(F_i)$ for $M(F_i)$, $1 \leq i \leq N$, in the definition of M . The same proof as [6], Prop. II.3, shows that T is dense in M , and by the continuity of μ to show $\mu(M) \cdot S(\nu, \mathfrak{B}) = (0)$ we need merely show $x \in \mathfrak{A}$, $z \in T \Rightarrow \mu(z)\lambda(x) = 0$ (the proof of $S(\nu, \mathfrak{B}) \cdot \mu(M) = (0)$ is symmetric). Clearly $zx \in T$, and μ and ν agree on T , so $\mu(z)\lambda(x) = \mu(z)(\nu(x) - \mu(x)) = \mu(z)\nu(x) - \mu(z)\mu(x) = \nu(zx) - \mu(zx) = 0$. That $\lambda|_M$ is a homomorphism follows from $\mu(M) \cdot S(\nu, \mathfrak{B}) = (0)$ and the arguments of Bade and Curtis ([1], p. 601).

The analogue of [1], Theorem 4.3d, will be stated but, once the definitions are made, the proofs precisely parallel the proofs given in [1], and so will be omitted. It should be noted, however, that the proofs carry over because, for $1 \leq i \leq N$, $C(X_i) \otimes B(\mathcal{H}_i)$ is actually the algebraic tensor product.

For $1 \leq i \leq N$, let $F_i = \{\omega_{i,k} \mid 1 \leq k \leq n_i\}$, and for each i , $1 \leq i \leq N$, choose functions $e_{i,k} \in C(X_i)$ such that $e_{i,k}$ is 1 in a neighborhood of $\omega_{i,k}$ and $e_{i,k}e_{i,j} = 0$, $k \neq j$. Let I_i denote the identity of $B(\mathcal{H}_i)$, and define $\lambda_{i,k}(x) = \lambda(\langle e_{i,k} \otimes I_i \rangle x)$ (note that this is equal to $\lambda(x \langle e_{i,k} \otimes I_i \rangle)$). Let $R_{i,k} = \overline{\lambda_{i,k}(\mathfrak{A})}$, let $M(\omega_{i,k})$ be all functions in $C(X_i)$ vanishing at $\omega_{i,k}$, and let $M_{i,k}$ be \mathfrak{A} with $C(X_i) \otimes B(\mathcal{H}_i)$ replaced in the direct sum by $M(\omega_{i,k}) \otimes B(\mathcal{H}_i)$.

PROPOSITION 1.2.

$$(a) \quad \lambda = \sum_{i=1}^N \sum_{k=1}^{n_i} \lambda_{i,k}$$

$$(b) \quad S(\nu, \mathfrak{B}) = \sum_{i=1}^N \sum_{k=1}^{n_i} \oplus R_{i,k},$$

the direct sum being topological.

$$(c) \quad (i, j) \neq (k, l) \Rightarrow R_{i,j} \cdot R_{k,l} = (0),$$

and

$$R_{i,k} \cdot \mu(M_{i,k}) = \mu(M_{i,k}) \cdot R_{i,k} = (0).$$

(d) The restriction of $\lambda_{i,k}$ to $M_{i,k}$ is a homomorphism.

It is possible to obtain a characterization of the ideal $S(\nu, \mathfrak{B})$ by examining the action of ν as related to the operator algebras $B(\mathcal{H}_i)$, rather than the function spaces $C(X_i)$. For $1 \leq i \leq N$, let e_i be the identity of $C(X_i)$, and let $\lambda_i(x) = \lambda(\langle e_i \otimes I_i \rangle x)$; then $\lambda(x) = \sum_{i=1}^N \lambda_i(x)$. Now

$$\begin{aligned}
& \mu(\langle e_j \otimes I_j \rangle) \lambda_i(x) \\
&= \mu(\langle e_j \otimes I_j \rangle) [\nu(\langle e_i \otimes I_i \rangle x) - \mu(\langle e_i \otimes I_i \rangle x)] \\
&= \nu(\langle e_j \otimes I_j \rangle \langle e_i \otimes I_i \rangle x) - \mu(\langle e_j \otimes I_j \rangle \langle e_i \otimes I_i \rangle x) \\
&= \delta_{ij} \lambda_i(x),
\end{aligned}$$

and if $i \neq j$ then

$$\lim_{n \rightarrow \infty} \lambda_i(x_n) = \lim_{n \rightarrow \infty} \lambda_j(y_n)$$

yields the fact that both these limits are zero, and consequently

$$S(\nu, \mathfrak{B}) = \sum_{i=1}^N \oplus \overline{\lambda_i(\mathfrak{A})},$$

a topological direct sum. Now each of these components will be characterized.

Fix n such that $1 \leq n \leq N$, and let $\{T_{i,j} \mid 1 \leq i, j \leq n\}$ be a system of matrix units for $B(\mathcal{H}_n)$, i.e., $T_{i,j} T_{k,l} = \delta_{jk} T_{i,l}$. Define, for $1 \leq i, j \leq n$, maps $\nu_{i,j}$, $\mu_{i,j}$, and $\gamma_{i,j}$ of $C(X_n)$ into \mathfrak{B} by $\nu_{i,j}(f) = \nu(\langle f \otimes T_{i,j} \rangle)$, $\mu_{i,j}(f) = \mu(\langle f \otimes T_{i,j} \rangle)$, and $\gamma_{i,j}(f) = \nu_{i,j}(f) - \mu_{i,j}(f)$. If

$$x = \left\langle \sum_{i,j=1}^n f_{i,j} \otimes T_{i,j} \right\rangle,$$

we can clearly write

$$\nu(x) = \sum_{i,j=1}^n \nu_{i,j}(f_{i,j});$$

similar assertions hold for $\mu(x)$ and $\lambda(x)$. All maps are linear, but the "off-diagonal" maps (those for which $i \neq j$) are not necessarily homomorphisms.

Computational procedures similar to those already employed will show

$$\mu(\langle e_n \otimes T_{k,l} \rangle) \gamma_{i,j}(f) = \delta_{il} \gamma_{k,j}(f)$$

and

$$\gamma_{i,j}(f) \mu(\langle e_n \otimes T_{k,l} \rangle) = \delta_{jk} \gamma_{i,l}(f),$$

so if

$$\lim_{m \rightarrow \infty} \gamma_{i,j}(f_m) = \lim_{m \rightarrow \infty} \gamma_{k,l}(g_m)$$

and $i \neq k$, left multiplication by $\mu(\langle e_n \otimes T_{i,i} \rangle)$ shows that

$$\lim_{m \rightarrow \infty} \gamma_{i,j}(f_m) = 0;$$

the same trick with right multiplication works if $j \neq l$, and so

$$\overline{\lambda_n(\mathfrak{A})} = \sum_{i,j=1}^n \bigoplus \overline{\gamma_{i,j}(\mathfrak{A})},$$

and this is a topological direct sum.

Since $T_{i,j} = T_{i,k}T_{k,j}$, we see that

$$\begin{aligned} \nu_{i,j}(fg) &= \nu(\langle fg \otimes T_{i,j} \rangle) \\ &= \nu(\langle f \otimes T_{i,k} \rangle \langle g \otimes T_{k,j} \rangle) = \nu_{i,k}(f)\nu_{k,j}(g); \end{aligned}$$

consequently $\nu_{i,i}$ is a homomorphism for $1 \leq i \leq n$ (let $j = k = i$) and so by [1], Th. 4.3b), $\overline{\gamma_{i,i}(\mathfrak{A})}$ is the Jacobson radical of $\overline{\nu_{i,i}(C(X_n))}$. Since $\mu(\langle e_n \otimes T_{i,j} \rangle)\gamma_{j,j}(f) = \gamma_{i,j}(f)$, it is clear that

$$\overline{\gamma_{i,j}(\mathfrak{A})} = \mu(\langle e_n \otimes T_{i,j} \rangle)\overline{\gamma_{j,j}(\mathfrak{A})}. \quad \text{This yields}$$

PROPOSITION 1.3. *$S(\nu, \mathfrak{B})$ is the direct sum of Jacobson radicals of commutative Banach algebras and “rotations” of these radicals.*

Note that $\nu_{i,j}(f) = \nu_{i,i}(f)\nu_{i,j}(e_n)$, and so the continuity of the $\nu_{i,j}$, and hence the continuity of ν , depends only on the continuity of the diagonal homomorphisms $\nu_{i,i}$. Coupling this fact with Theorem 4.5 of [1], we observe that if all the Jacobson radicals of the closures of the images of the diagonal homomorphisms are nil ideals, then the homomorphism is continuous.

2. Homomorphisms of B^* -algebras. Let \mathfrak{A} be a B^* -algebra, and let $\nu: \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism, with $S(\nu, \mathfrak{B})$ defined as in §1.

DEFINITION 2.1.

$$\begin{aligned} \mathcal{I}_L &= \{x \in \mathfrak{A} \mid \nu(x) \cdot S(\nu, \mathfrak{B}) = (0)\}, \\ \mathcal{I}_R &= \{x \in \mathfrak{A} \mid S(\nu, \mathfrak{B}) \cdot \nu(x) = (0)\}. \end{aligned}$$

DEFINITION 2.2.

$$\begin{aligned} \mathcal{I}_L &= \{x \in \mathfrak{A} \mid \sup_{\|z\| \leq 1} \|\nu(xz)\| < \infty\}, \\ \mathcal{I}_R &= \{x \in \mathfrak{A} \mid \sup_{\|z\| \leq 1} \|\nu(zx)\| < \infty\}, \\ \mathcal{I} &= \mathcal{I}_L \cap \mathcal{I}_R. \end{aligned}$$

$\mathcal{I}_L, \mathcal{I}_R, \mathcal{I}_L, \mathcal{I}_R$, and \mathcal{I} are all two-sided ideals in \mathfrak{A} (see [4] and [6]), and in a recent paper [4] Johnson has shown that $\overline{\mathcal{I}_L}$ is a cofinite ideal in \mathfrak{A} , and observes that, if one could show $\nu|_{\overline{\mathcal{I}_L}}$ is continuous, one would have a direct extension of the Bade-Curtis

theory to arbitrary B^* -algebras. An examination of this problem, coupled with an analysis of these ideals, constitutes the body of this section.

We first note that $\mathcal{J}_L \subset \mathcal{J}_L$. For, if $x \notin \mathcal{J}_L$ then there is an $s \in S(\nu, \mathfrak{B})$ such that $\nu(x)s \neq 0$, and consequently $\exists \{x_n\} \subset \mathfrak{A}$ such that $x_n \rightarrow 0$, $\nu(x_n) \rightarrow s$, and so $\nu(xx_n) \rightarrow \nu(x)s \neq 0$. Given $M > 0$, choose x_n such that

$$\|x_n\| \leq \frac{\|\nu(x)s\|}{2M}, \quad \|\nu(xx_n)\| > \frac{1}{2} \|\nu(x)s\|.$$

Then

$$\frac{x_n}{\|x_n\|}$$

has norm one and

$$\left\| \nu\left(x \frac{x_n}{\|x_n\|}\right) \right\| > M,$$

and so $x \notin \mathcal{J}_L$. Similarly $\mathcal{J}_R \subset \mathcal{J}_R$.

Repeated use throughout this section will be made of the following lemma and its corollaries.

LEMMA 2.1. *Let $\{f_n\}$, $\{g_n\}$ be sequences from \mathfrak{A} such that $m \neq n \Rightarrow g_m g_n = 0$, $g_n f_m = 0$. Then there is an integer N such that $n \geq N \Rightarrow g_n f_n \in \mathcal{J}_R$.*

Proof. Suppose not, and renumber to obtain a sequence such that $g_n f_n \notin I_R$ for any n . Then for each n choose $x_n \in \mathfrak{A}$ such that $\|x_n\| \leq 1$,

$$\|\nu(x_n g_n f_n)\| > n 2^n \|g_n\| \|\nu(f_n)\|.$$

Let

$$x = \sum_{k=1}^{\infty} (1/2^k \|g_k\|) x_k g_k;$$

then clearly $x \in \mathfrak{A}$. We also have

$$x f_n = \sum_{k=1}^{\infty} (1/2^k \|g_k\|) x_k g_k f_n = x_n g_n f_n / (2^n \|g_n\|),$$

and so

$$\begin{aligned} \|\nu(x)\| \|\nu(f_n)\| &\geq \|\nu(x f_n)\| \\ &= \|\nu(x_n g_n f_n)\| / 2^n \|g_n\| > n \|\nu(f_n)\|, \end{aligned}$$

which implies $\|\nu(x)\| > n$, a contradiction.

COROLLARY 2.1.1. *If $\{g_n\}$, $\{f_n\} \subset \mathfrak{A}$ satisfy $g_m g_n = 0$, $g_n f_n = f_n$, then $\exists N$ such that $n \geq N \Rightarrow f_n \in \mathcal{J}_R$.*

COROLLARY 2.1.2. *If $\{f_n\} \in \mathfrak{A}$ satisfies $f_m f_n = 0$, then $\exists N$ such that $n \geq N \Rightarrow f_n^2 \in I_R$.*

COROLLARY 2.1.3. *If $\{f_n\}, \{g_n\} \subset \mathfrak{A}$ satisfy $g_m g_n = 0, f_m g_n = 0$, then $\exists N$ such that $n \geq N \Rightarrow f_n g_n \in I_L$.*

We can now combine these results with those of Johnson ([4], Th. 2.1) to see that, if \mathfrak{A} is a B^* -algebra, $\overline{\mathcal{J}}$ is a cofinite ideal. The advantage of using $\overline{\mathcal{J}}$ can be seen from the following.

PROPOSITION 2.1. *Let $\nu: \mathfrak{A} \rightarrow \mathfrak{B}$ be a Banach algebra homomorphism, and let \mathfrak{U} be a closed linear subspace of \mathcal{J} . Then*

$$\sup \{ \|\nu(xy)\| \mid x, y \in \mathfrak{U}, \|x\| \leq 1, \|y\| \leq 1 \} < \infty .$$

Proof. For $z \in \mathfrak{A}$, let L_z and R_z map \mathfrak{A} into \mathfrak{B} and be defined by $L_z(x) = \nu(zx)$, $R_z(x) = \nu(xz)$; these are clearly linear. If $z \in \mathcal{J}$, then both L_z and R_z are continuous. For, if $x_n \rightarrow 0$ and $L_z(x_n) \rightarrow 0$, we can assume $\|L_z(x_n)\| \geq \delta > 0$. Given $M > 0$, choose x_n such that

$$\|x_n\| \leq \frac{\delta}{M} ,$$

then

$$\left\| \frac{M}{\delta} x_n \right\| \leq 1 , \quad \left\| L_z \left(\frac{M}{\delta} x_n \right) \right\| \geq M ;$$

since this can be done for any M it contradicts $z \in \mathcal{J}$. Now, for each $x \in \mathfrak{U}$,

$$\begin{aligned} \sup \{ \|L_z(x)\| \mid z \in \mathfrak{U}, \|z\| \leq 1 \} &= \sup \{ \|\nu(zx)\| \mid z \in \mathfrak{U}, \|z\| \leq 1 \} \\ &\leq \sup \{ \|\nu(zx)\| \mid z \in \mathfrak{A}, \|z\| \leq 1 \} < \infty \end{aligned}$$

since $x \in \mathcal{J}$. By the Uniform Boundedness Principle ([3], 2.3.21)

$$\sup \{ \|L_z\| \mid z \in \mathfrak{U}, \|z\| \leq 1 \} < \infty$$

and so

$$\sup \{ \|\nu(zx)\| \mid z, x \in \mathfrak{U}, \|z\| \leq 1, \|x\| \leq 1 \} < \infty$$

completing the proof.

PROPOSITION 2.2. *Let \mathfrak{A} be a C^* -algebra, and let $\mathfrak{U} \subseteq \mathcal{J}$ be a closed two-sided ideal. Then $\nu|_{\mathfrak{U}}$ is continuous.*

Proof. Let $U \in \mathfrak{U}$, and recall that \mathfrak{U} is a $*$ -ideal. Use the polar decomposition to write $U = TP$, where T is a partial isometry (hence $\|T\| = 1$) and P is a positive operator satisfying $P^2 = U^*U$. Assume $\|U\| = 1$, then since P is self-adjoint, $\|P\|^2 = \|P^*P\| = \|P^2\| =$

$\|U^*U\| = \|U\|^2 = 1$, so $\|P\| = 1$. Since P is self-adjoint, it has a square root $Q \in \mathfrak{U}$, so we can write $U = (TQ)Q$, where $TQ, Q \in \mathfrak{U}$, $\|TQ\| \leq \|T\|\|Q\| \leq 1, \|Q\| \leq 1$. So, by Proposition 2.1,

$$\begin{aligned} & \sup \{ \|\nu(U)\| \mid U \in \mathfrak{U}, \|U\| \leq 1 \} \\ & \leq \sup \{ \|\nu(xy)\| \mid x, y \in \mathfrak{U}, \|x\| \leq 1, \|y\| \leq 1 \} < \infty, \end{aligned}$$

and so $\nu|_{\mathfrak{U}}$ is continuous.

If \mathfrak{U} is a commutative B^* -algebra, Proposition 2.2 shows that, if N is a closed neighborhood of the Bade-Curtis singularity set, ν is continuous on the ideal of all functions vanishing on N , and Proposition 2.2 can be regarded as the analogue for B^* -algebras of that theorem, especially in view of the remarks following Corollary 2.1.3. However, it appears to be a difficult problem to obtain the full strength of the Bade-Curtis results using these methods, but if a method is found there is a good chance that it would generalize the Bade-Curtis results to arbitrary B^* -algebras.

We now turn our attention to $C(X)$, where X is a compact Hausdorff space. The notation of § 1 applies.

PROPOSITION 2.3. $T(F) \subseteq \mathcal{J}$, and if \mathcal{J} is closed, ν is continuous.

Proof. Let f vanish on a neighborhood of F . If $f \notin \mathcal{J}$, $\exists \{g_n\} \in C(X)$ such that $\|g_n\| \leq 1, \|\nu(fg_n)\| \geq n^2$. Let $h_n = 1/n g_n$, then $h_n f \rightarrow 0$, and since ν is continuous on $T(F)$, $\nu(h_n f) \rightarrow 0$. But

$$\|\nu(h_n f)\| = \frac{1}{n} \|\nu(g_n f)\| \geq n,$$

a contradiction.

If \mathcal{J} is closed, $M(F) = \overline{T(F)} \subseteq \mathcal{J}$, and by Proposition 2.2, $\nu|_{M(F)}$ is continuous. Using the technique of Theorem 4.1 of [1], ν is continuous.

Since $T(F) \subseteq \mathcal{J}$ and, if K denotes the kernel of ν , $\bar{K} \cap T(F) = K \cap T(F)$ ([7], 2.3), one might wish to show that $\bar{K} \cap \mathcal{J} = K$ (clearly $K \subseteq \mathcal{J}$). If $f \in \bar{K} \cap \mathcal{J}$, then $g_n \rightarrow 0 \Rightarrow \nu(g_n f) \rightarrow 0$. Let $g \in M(F)$, and choose a sequence $\{g_n\}$ from $T(F)$ such that $g_n \rightarrow g$. Then $g_n f \in \bar{K} \cap T(F) \subseteq K$, and so

$$\nu(gf) = \lim_{n \rightarrow \infty} \nu(g_n f) = 0.$$

So $M(F) \cdot (\bar{K} \cap \mathcal{J}) \subseteq K$.

If $\mathfrak{X} = C(X)$, Corollary 2.1.2 can be strengthened so the conclusion is $\exists N$ such that $n \geq N \Rightarrow f_n \in \mathcal{J}$. If this integer N is independent

of the sequence $\{f_n\}$, then the homomorphism is continuous, if X is such that every point is a G_δ . We first note that, if $\{E_n \mid n = 1, 2, \dots\}$ is a disjoint sequence of open sets, then $n \geq N, f(E'_n) = 0 \Rightarrow f \in \mathcal{J}$; this is a clear consequence of Corollary 2.1.2. The goal will be to show that, if N is independent of sequence, then $M(F) \subseteq \mathcal{J}$, as in Proposition 2.3 this will show ν is continuous. Choose open sets $E, G \subseteq X$ such that $\bar{E} \cap \bar{G} = F$, and let $f \in M(F)$. Let

$$A_k = \left\{ x \in X \mid |f(x)| \geq \frac{1}{k} \right\},$$

and let $B_k = A_k \cap \bar{G}$; then B_k is closed and disjoint from \bar{E} for all k . By Urysohn's Lemma, choose a function g_k such that $0 \leq g_k \leq 1$, $g_k(\bar{E}) = 1$, $g_k(B_k) = 0$. We assert that $\{g_n f \mid n = 1, 2, \dots\}$ is Cauchy. Assume $n > m$, and look at $\|g_n f - g_m f\|$. This value is the maximum of the supremums of $|g_n f(x) - g_m f(x)|$ on the sets \bar{E}, B_m , and $K_m = X \sim (B_m \cup \bar{E})$. This supremum is clearly 0 on \bar{E} (since $g_n(\bar{E}) = g_m(\bar{E}) = 1$) and on B_m (since $n > m \Rightarrow B_m \subseteq B_n$), and clearly

$$\sup_{x \in K_m} |g_n f(x) - g_m f(x)| \leq \frac{1}{n} + \frac{1}{m} < \frac{2}{m},$$

so the sequence is Cauchy, and there is an $h \in C(X)$ such that $\|g_n f - h\| \rightarrow 0$. $h(\bar{E}) = f(\bar{E})$, since $g_n(\bar{E}) = 1$ for all n . If $x \in \bar{G}$ and $|f(x)| > 0$, there is an integer K such that $k \geq K \Rightarrow x \in A_k \Rightarrow x \in B_k \Rightarrow g_k f(x) = 0$; if $f(x) = 0$ $g_k f(x) = 0$ for all k , and so $h(\bar{G}) = 0$.

Now choose sequences of disjoint open sets $\{E_n\}, \{G_n\}$ (the E_n are not necessarily disjoint from the G_n) such that $F \subseteq \bar{E}_n \cap \bar{G}_n, \bar{E} \supseteq E'_n$, and $\bar{G} \supseteq G'_n$. If $g \in C(X)$, $g(G'_n) = 0 \Rightarrow g \in \mathcal{J}$, or $g(E'_n) = 0 \Rightarrow g \in \mathcal{J}$, so $h(\bar{G}) = 0 \Rightarrow h \in \mathcal{J}$; similarly $(h - f)(\bar{E}) = 0 \Rightarrow h - f \in \mathcal{J}$, so $f = h + (f - h) \in \mathcal{J}$. Thus $M(F) \subseteq \mathcal{J}$, completing the proof. A similar idea also works for von Neumann algebras by reducing it to a consideration of $\varphi_i: C(X) \rightarrow \mathfrak{B}$ defined by $\varphi_i(f) = \nu(\langle f \otimes I_i \rangle)$.

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