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This paper considers three transforms of a complex series Σa_n: namely, (1) Aitken's δ^2-transform Σb_n, (2) Lubkin's W-transform Σc_n, and (3) a closely related transform Σd_n which the author calls the $W1$-transform and for which $\sum_0^n d_k = \sum_0^{n+1} c_k$. If $a_{n-1} \neq 0$, set $r_n = a_n/a_{n-1}$. If, moreover, Σa_n converges, define $T_n = (a_n + a_{n+1} + \cdots)/a_{n-1}$ and let $MR(\Sigma a_n)$ be the class of all series converging more rapidly to the sum $S = \Sigma a_n$ than Σa_n. Some of the results proven in this paper are as follows:

(1) If $b_n/a_n \to 0$, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Sigma c_n \in MR(\Sigma a_n)$, and (iii) $\Sigma d_n \in MR(\Sigma a_n)$ are equivalent.

(2) $\Sigma b_n \in MR(\Sigma a_n)$ if and only if $\Delta T_n \to 0$.

(3) If $|r_n| \leqq \rho < 1$ for all sufficiently large n, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Delta r_n \to 0$, and (iii) $b_n/a_n \to 0$ are equivalent.

Samuel Lubkin has given several sufficient conditions for $\Sigma b_n \in MR(\Sigma a_n)$ in case Σa_n is a real series. The third result above contains a generalization of one of his results to the complex plane while relaxing some of his hypothesis.

The following results on complex products are also proven:

(4) If the sequence $\{1/a_n - 1/a_{n-1}\}$ is bounded, then the product $\Pi_0^\infty (1 + a_n)$ diverges.

(5) Suppose that $|r_n| \leqq \rho < 1$ for all sufficiently large n and $a_n \neq -1$ for all n. Then a necessary and sufficient condition for the δ^2-transform to accelerate the convergence of the infinite product $\Pi_0^\infty (1 + a_n)$ is that $\Delta r_n \to 0$.

The notations and definitions set forth in Tucker [2] will be used in this paper. In particular, $S_n = a_0 + a_1 + \cdots + a_n$, $\Sigma a_n = \sum_0^\infty a_n$, and $S = \Sigma a_n$ if Σa_n is convergent. Given a second series $\Sigma a'_n$ we use the notation $S'_n = a'_0 + \cdots + a'_n$, $r'_n = a'_n/a'_{n-1}$ for $a'_{n-1} \neq 0$, $S' = \Sigma a'_n$ and $T'_n = (S' - S'_{n-1})/a'_{n-1}$ for $a'_{n-1} \neq 0$. Likewise, given a "transform sequence" $\{\alpha_n\}$, α_n complex, we set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$ for $n \geqq 0$, $a_{\alpha 0} = S_{\alpha 0} = a_0 + a_1\alpha_1$, and $a_{\alpha n} = S_{\alpha n} - S_{\alpha(n-1)}$ for $n \geqq 1$.

The transform sequences associated with the δ^2, W, and $W1$ transforms are defined respectively as follows:

(i) $\alpha_n = 1/(1 - r_n)$, $n \geqq 1$,

(ii) $\alpha_1 = -a_0/a_1$; $\alpha_n = (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$, $n \geqq 2$,

(iii) $\alpha_n = (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$, $n \geqq 1$.

Whenever division by zero occurs in (i), we set $\alpha_n = 0$. We do likewise for (ii) and (iii). As in Tucker [2], we retain the notation

$\{\delta_n\}$ for the δ^2-transform sequence, and if "$*$" denotes any relation, the notation "$*.$" means that $*$ holds for all sufficiently large n and "$*:$" means that $*$ holds for infinitely many positive integers n.

In what follows, the author is generally interested in the interrelationships between the conditions (1) $\Sigma b_n \in MR(\Sigma a_n)$, (2) $\Sigma c_n \in MR(\Sigma a_n)$, (3) $\Sigma d_n \in MR(\Sigma a_n)$, (4) $b_n/a_n \to 0$, (5) $\Delta T_n \to 0$, (6) $\Delta r_n \to 0$, (7) $|r_n| \leqq . B$ for some B, and (8) $0 < B \leqq . |1 - r_n|$ for some B. Also, the notation Σb_n, Σc_n and Σd_n specified in the first paragraph for the respective δ^2, W and $W1$ transforms will not be used in what follows. Instead, the appropriate $\Sigma a_{\delta n}$ or $\Sigma a_{\alpha n}$ notation will be employed.

The following two theorems, the second in particular, are helpful when investigating acceleration.

THEOREM 1. *Suppose that Σa_n is a complex series, $\{b_n\}$ is a complex sequence, and $\Sigma a'_n$ is a series with partial sums $S'_n = . S_n + b_{n+1}$. Then $\Sigma a'_n \in MR(\Sigma a_n)$ if and only if $b_{n+1} \sim S - S_n \to 0$.*

Proof. If either condition holds, then

$$S - S_n = . S - S'_n + b_{n+1} \neq . 0 ,$$

so that $b_{n+1}/(S-S_n) + (S-S'_n)/(S-S_n) = . 1$. Thus $(S-S'_n)/(S-S_n) \to 0$ and $S - S_n \to 0$, if and only if, $b_{n+1}/(S - S_n) \to 1$ and $S - S_n \to 0$; but this is equivalent to $b_{n+1} \sim S - S_n \to 0$.

From Theorem 1, we see that the class of all sequences $\{c_n\}$ such that $\Sigma a'_n \in MR(\Sigma a_n)$, where $S'_n = S_n + c_{n+1}$, is completely determined by one such sequence $\{b_n\}$; the required condition being that $c_n \sim b_n$. Similarly, we now show that if $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, then $\Sigma a_{\beta n} \in MR(\Sigma a_n)$, if and only if $\beta_n \sim \alpha_n$.

THEOREM 2. *Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $\Sigma a_{\beta n} \in MR(\Sigma a_n)$ if and only if $\beta_n \sim \alpha_n$.*

Proof. From Theorem 1, $a_{n+1}\alpha_{n+1} \sim S - S_n \to 0$. Hence, from Theorem 1, $\Sigma a_{\beta n} \in MR(\Sigma a_n)$ if and only if $a_{n+1}\beta_{n+1} \sim S - S_n$, and this is equivalent to $a_{n+1}\beta_{n+1} \sim a_{n+1}\alpha_{n+1}$, that is, $\beta_{n+1} \sim \alpha_{n+1}$.

LEMMA 3. *If $(1 - r_n)(1 - r_{n+1}) \neq 0$, then $a_{\delta n}/a_n = 1/(1 - r_{n+1}) - 1/(1 - r_n) = r_{n+1}/(1 - r_{n+1}) - r_n/(1 - r_n) = (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$.*

Proof. Since $r_n \neq 1$ and $r_{n+1} \neq 1$, we have $\delta_n = 1/(1 - r_n)$ and $\delta_{n+1} = 1/(1 - r_{n+1})$. Thus, $a_{\delta n}/a_n = (a_n + a_{n+1}\delta_{n+1} - a_n\delta_n)/a_n = 1 + r_{n+1}\delta_{n+1} - \delta_n = r_{n+1}/(1 - r_{n+1}) + 1 - 1/(1 - r_n) = r_{n+1}/(1 - r_{n+1}) - r_n/(1 - r_n) = [r_{n+1}(1 - r_n) -$

$r_n(1-r_{n+1})]/(1-r_n)(1-r_{n+1}) = (r_{n+1}-r_n)/(1-r_n)(1-r_{n+1}) = 1/(1-r_{n+1}) - 1/(1-r_n)$.

We now establish a relationship between the δ^2-transform and the $W1$-transform.

THEOREM 4. *Suppose that* $a_{\delta n}/a_n \to 0$. *Then* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$.

Proof. Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. From Lemma 3,

$$\begin{aligned}1 - 2r_{n+1} + r_n r_{n+1} &=. (1-r_n)(1-r_{n+1}) - (r_{n+1}-r_n)\\ &=. (1-r_n)(1-r_{n+1}) \cdot [1-(r_{n+1}-r_n)/(1-r_n)(1-r_{n+1})]\\ &=. (1-r_n)(1-r_{n+1})(1-a_{\delta n}/a_n) \neq. 0 .\end{aligned}$$

Hence, $\alpha_n/\delta_n =. (1-r_n)(1-r_{n+1})/(1-2r_{n+1}+r_n r_{n+1}) =. 1/(1-a_{\delta n}/a_n) \to 1$. From Theorem 2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $r_n \neq. 1$, so that

$$\alpha_n/\delta_n =. 1/(1-a_{\delta n}/a_n) \to 1$$

and, from Theorem 2, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The same type of relationship is now established between the δ^2-transform and the W-transform.

THEOREM 5. *Suppose that* $a_{\delta n}/a_n \to 0$. *Then* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1-r_{n-1})/(1-2r_n+r_{n-1}r_n)$.

Proof. Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. As in the proof of Theorem 4,

$$1 - 2r_n + r_{n-1}r_n =. (1-r_{n-1})(1-r_n)[1-a_{\delta(n-1)}/a_{n-1}] \neq. 0 .$$

Hence,

$$\alpha_n/\delta_n =. (1-r_{n-1})(1-r_n)/(1-2r_n+r_{n-1}r_n) =. 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1 .$$

From Theorem 2, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

Suppose that $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$. Then $r_n \neq. 1$, and thus

$$\alpha_n/\delta_n =. 1/(1-a_{\delta(n-1)}/a_{n-1}) \to 1 .$$

From Theorem 2, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The next theorem helps to establish the significance of the quantities T_n when dealing with acceleration in general.

THEOREM 6. $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, $\alpha_n \sim T_n/r_n$, *and* $\alpha_n \sim 1 + T_{n+1}$ *are equivalent*.

Proof. From Theorem 1, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if and only if $a_{n+1}\alpha_{n+1} \sim S - S_n \to 0$; and this is equivalent to $\alpha_{n+1} \sim (S - S_n)/a_{n+1} = T_{n+1}/r_{n+1}$. Moreover, $\alpha_n \sim T_n/r_n$ is equivalent to $\alpha_n \sim 1 + T_{n+1}$, since $T_n/r_n = 1 + T_{n+1}$.

We now establish a useful algebraic expression for $(S - S_{\delta(n-1)})/(S - S_{n-1})$ in terms of ΔT_n.

LEMMA 7. *If Σa_n is a convergent series and n is a positive integer such that $T_{n+1} - T_n \neq -1$, then*

$$(S - S_{\delta(n-1)})/(S - S_{n-1}) = (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) .$$

Proof. From $(1 - r_n)(1 + T_{n+1}) = 1 + T_{n+1} - T_n \neq 0$, $T_{n+1} \neq -1$ and $r_n \neq 1$. Thus $S - S_{n-1} = a_n(1 + T_{n+1}) \neq 0$. We then have

$$\begin{aligned}(S - S_{\delta(n-1)})/(S - S_{n-1}) &= (S - S_{n-1} - a_n\delta_n)/(S - S_{n-1}) \\ &= 1 - a_n\delta_n/(S - S_{n-1}) \\ &= 1 - \frac{a_n}{S - S_{n-1}}\,\frac{1}{1 - r_n} = 1 - \frac{1}{T_n}\,\frac{r_n}{1 - r_n} \\ &= 1 - \frac{T_n/(1 + T_{n+1})}{1 - T_n/(1 + T_{n+1})}\,\frac{1}{T_n} \\ &= 1 - 1/(1 + T_{n+1} - T_n) = (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) .\end{aligned}$$

We now establish necessary and sufficient conditions for the δ^2-process to accelerate the convergence of a convergent series Σa_n.

THEOREM 8. *$\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $T_{n+1} - T_n \to 0$.*

1*st Proof.* From Theorem 6, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $\delta_n \sim 1 + T_{n+1}$, and this is equivalent to $(1 + T_{n+1})(1 - r_n) \to 1$, since $\delta_n =. 1/(1 - r_n)$. Finally, $(1 + T_{n+1})(1 - r_n) \to 1$ if and only if $T_{n+1} - T_n \to 0$, since $T_{n+1} - T_n =. (1 + T_{n+1})(1 - r_n) - 1$.

2*nd Proof.* If $T_{n+1} - T_n \to 0$, then $T_{n+1} - T_n \neq. -1$. Thus, from Lemma 7, $(S - S_{\delta(n-1)})/(S - S_{n-1}) =. (T_{n+1} - T_n)/(1 + T_{n+1} - T_n) \to 0$. Conversely, suppose that $(S - S_{\delta(n-1)})/(S - S_{n-1}) \to 0$. Then $a_n \neq. 0$ and $r_n \neq. 1$, since $\delta_n \neq. 0$. We must have $1 + T_{n+1} - T_n \neq. 0$, since otherwise $(1 - r_n)(T_n/r_n) =. 1 + T_{n+1} - T_n =: 0$, and $S - S_{n-1} =: 0$; a contradiction. From Lemma 7, $(T_{n+1} - T_n)/(1 + T_{n+1} - T_n) =. (S - S_{\delta(n-1)})/(S - S_{n-1}) \to 0$, and thus $T_{n+1} - T_n \to 0$.

The preceding theorem immediately yields the corollary, also proven in Tucker [2], that the convergence of $\{T_n\}$ imples $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

LEMMA 9. *If Σa_n is a convergent series and n is a positive integer such that $a_{n-1}a_na_{n+1} \neq 0$, then*

$$r_{n+1} - r_n = (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) \\ - (T_{n+2} - T_{n+1})(1 - r_n) + (T_{n+1} - T_n)(1 - r_{n+1}) .$$

Proof. We have

$$(1 - r_n)(1 + T_{n+1}) = 1 - r_n + T_{n+1} - r_nT_{n+1} \\ = 1 + T_{n+1} - r_n(1 + T_{n+1}) = 1 + T_{n+1} - T_n ,$$

so that

$$T_{n+1} - T_n = (1 - r_n)(1 + T_{n+1}) - 1 .$$

Similarly,

$$T_{n+2} - T_{n+1} = (1 - r_{n+1})(1 + T_{n+2}) - 1 .$$

Thus,

$$\begin{aligned}(T_{n+2} &- T_{n+1})(1 - r_n)(1 - r_{n+1}) - (T_{n+2} - T_{n+1})(1 - r_n) \\ &+ (T_{n+1} - T_n)(1 - r_{n+1}) = (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) \\ &- (1 - r_n)[(1 - r_{n+1})(1 + T_{n+2}) - 1] \\ &+ (1 - r_{n+1})[(1 - r_n)(1 + T_{n+1}) - 1] \\ &= (T_{n+2} - T_{n+1})(1 - r_n)(1 - r_{n+1}) + (1 - r_n) \\ &- (1 - r_n)(1 - r_{n+1})(1 + T_{n+2}) - (1 - r_{n+1}) \\ &+ (1-r_n)(1-r_{n+1})(1+T_{n+1}) = (1-r_n)(1-r_{n+1})[(T_{n+2}-T_{n+1}) \\ &- (1 + T_{n+2}) + (1 + T_{n+1})] + r_{n+1} - r_n = r_{n+1} - r_n .\end{aligned}$$

LEMMA 10. *If Σa_n is a convergent series and n is a positive integer such that $(1 - r_n)(1 - r_{n+1})a_{n+1} \neq 0$, then $a_{\delta n}/a_n = (T_{n+2} - T_{n+1}) - (T_{n+2} - T_{n+1})/(1 - r_{n+1}) + (T_{n+1} - T_n)/(1 - r_n)$.*

Proof. We have $a_{n-1}a_na_{n+1} \neq 0$, and

$$a_{\delta n}/a_n = (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$$

according to Lemma 3. We now apply Lemma 9.

LEMMA 11. *If $a_{\delta n} \in MR(\Sigma a_n)$ and $0 < B \leqq . \, |1 - r_n|$ for some number B, then $a_{\delta n}/a_n \to 0$.*

Poof. From Theorem 8, $T_{n+1} - T_n \to 0$. Using Lemma 10 and $0 < B \leqq . \, |1 - r_n|$, it is obvious that $a_{\delta n}/a_n \to 0$.

THEOREM 12. *Suppose that $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ and $0 < B \leqq . \, |1 - r_n|$.*

Then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1} r_n)$.

Proof. From Lemma 11, $a_{\delta n}/a_n \to 0$. We now apply Theorem 4, if $\alpha_n =. (1 - r_{n+1})/(1-2r_{n+1}+r_n r_{n+1})$; or Theorem 5, if $\alpha_n =. (1-r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$.

THEOREM 13. *If* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *and* $|r_n| \leqq. B$ *for some number* B, *then* $r_{n+1} - r_n \to 0$.

Proof. From Theorem 8, Lemma 9, and $|r_n| \leqq. B$, it is obvious that $r_{n+1} - r_n \to 0$.

The following theorem gives simple necessary and sufficient conditions for the δ^2-transform to accelerate convergence in the complex plane under the fairly general condition that $|r_n| \leqq. \rho < 1$. In addition, it generalizes the result on acceleration contained in Theorem 2 of Lubkin [1].

THEOREM 14. *Suppose that* $|r_n| \leqq. \rho < 1$ *for some number* ρ. *Then a necessary and sufficient condition that* $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *is that* $r_{n+1} - r_n \to 0$.

Proof. Since $|r_n| \leqq. \rho < 1$, Σa_n converges. The necessity follows from Theorem 13. For the sufficiency, let $\varepsilon > 0$. Since $r_{n+1}-r_n\to 0$, $|r_{n+1} - r_n| \leqq. \varepsilon$. Consequently,

$$\begin{aligned} |T_{n+1} - T_n| =. &|(r_{n+1} - r_n) + r_{n+1}(r_{n+2} - r_n) + r_{n+1}r_{n+2}(r_{n+3} - r_n) \\ &+ \cdots + (r_{n+1} \cdots r_{n+k-1})(r_{n+k} - r_n) + \cdots | \leqq. |r_{n+1} - r_n| \\ &+ |r_{n+1}|\,|r_{n+2} - r_n| + \cdots + |r_{n+1} \cdots r_{n+k-1}|\,|r_{n+k} - r_n| \\ &+ \cdots \leqq. \varepsilon + 2\varepsilon |r_{n+1}| + \cdots + k\varepsilon |r_{n+1} \cdots r_{n+k-1}| \\ &+ \cdots \leqq. \varepsilon[1 + 2\rho + 3\rho^2 + \cdots + k\rho^{k-1} + \cdots] = \varepsilon/(1 - \rho^2) . \end{aligned}$$

Hence $T_{n+1} - T_n \to 0$, and thus, from Theorem 8, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$.

The preceding theorem yields a simple proof of acceleration in a punctured disk in the complex place for certain power series as is now seen.

COROLLARY 15. *Suppose that* $|r_n| \leqq. \rho < 1$ *for some number* ρ, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ *and* $a'_n = a_n z^n$ *for every* n. *Then* $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, *for each complex number* z *satisfying* $0 < |z| < 1/\rho$.

Proof. From Theorem 14, $r_{n+1} - r_n \to 0$. Let z be any complex number such that $0 < |z| < 1/\rho$. Then $|r'_n| =. |r_n z| \leqq. \rho |z| < 1$ and $r'_{n+1} - r'_n =. r_{n+1}z - r_n z =. z(r_{n+1} - r_n) \to 0$. Thus $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, according to Theorem 14.

COROLLARY 16. *Suppose that* $|r_n| \leqq. \rho < 1$ *for some number* ρ, $r_{n+1} - r_n \to 0$ *and* $a'_n = a_n z^n$ *for every* n. *Then* $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$, *for each complex number* z *satisfying* $0 < |z| < 1/\rho$.

Proof. From Theorem 14, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. We now apply Corollary 15.

LEMMA 17. *If* $0 < A \leqq. |1 - r_n| \leqq. B$, *then* $a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$, *and* $a_{\delta n}/a_n \to 0$ *if and only if* $r_{n+1} - r_n \to 0$.

Proof. Since $0 < A \leqq. |1 - r_n| \leqq. B$, $0 < A^2 \leqq. |(1-r_n)(1-r_{n+1})| \leqq. B^2$. Hence from Lemma 3, $a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1})$. Thus from $0 < A^2 \leqq. |(1 - r_n)(1 - r_{n+1})| \leqq. B^2$, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$.

LEMMA 18. *If* $|r_n| \leqq. \rho < 1$, *then*

$$a_{\delta n}/a_n =. (r_{n+1} - r_n)/(1 - r_n)(1 - r_{n+1}) ,$$

and $a_{\delta n}/a_n \to 0$ *if and only if* $r_{n+1} - r_n \to 0$.

Proof. From $|r_n| \leqq. \rho < 1$, $0 < 1 - \rho \leqq. |1 - r_n| \leqq. 2$. We now apply Lemma 17.

THEOREM 19. *Suppose that* $|r_n| \leqq. \rho < 1$. *Then* $a_{\delta n} \in MR(\Sigma a_n)$ *if and only if* $a_{\delta n}/a_n \to 0$.

Proof. From Lemma 18, $a_{\delta n}/a_n \to 0$ if and only if $r_{n+1} - r_n \to 0$. From Theorem 14, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $r_{n+1} - r_n \to 0$. Consequently, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$ if and only if $a_{\delta n}/a_n \to 0$.

THEOREM 20. *If* $|r_n| \leqq. \rho < 1$ *and* $a_{\delta n}/a_n \to 0$, *then* $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$.

Proof. From Theorem 19, $\Sigma a_{\delta n} \in MR(\Sigma a_n)$. From Theorem 4, $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$ if $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$. If

$$\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n) ,$$

we may apply Theorem 5 to obtain $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$.

THEOREM 21. *If*

$$|r_n| \leqq. \rho < 1 \quad \textit{and} \quad r_{n+1} - r_n \to 0 ,$$

then $\Sigma a_{\alpha n} \in MR(\Sigma a_n)$, *where* $\alpha_n =. (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$ *or* $\alpha_n =. (1 - r_{n-1})/(1 - 2r_n + r_{n-1} r_n)$.

Proof. From Lemma 18, $a_{\delta n}/a_n \rightarrow 0$. We now apply Theorem 20.

In Tucker [2] it was proven in Theorem 3.7 that if $a'_n/a_n \rightarrow 0$, $|r_n| \leqq. \rho_1 < 1/2$ and $|r'_n| \leqq. \rho_2 < 1$, then $\Sigma a'_n$ converges more rapidly than Σa_n. Furthermore, it was shown there in Counterexample 3.8 that the replacement of "1/2" by any larger number produced in invalid result. We now turn to our next theorem which shows that "1/2" may be replaced by "1" under the additional hypothesis that $\Delta r_n \rightarrow 0$.

THEOREM 22. *If*

$$a'_n/a_n \rightarrow 0, |r_n| \leqq. \rho_1 < 1, |r'_n| \leqq. \rho_2 < 1$$

and $\Delta r_n \rightarrow 0$, *then* $\Sigma a'_n$ *converges more rapidly than* Σa_n.

Proof. From Theorems 8 and 14, $\Delta T_n \rightarrow 0$. Also $|1 + T'_{n+1}| \leqq. 1/(1 - \rho_2)$. Thus,

$$\frac{|S' - S'_{n-1}|}{|S - S_{n-1}|} =. \frac{|a'_n|}{|a_n|} \frac{|T'_n/r'_n|}{|T_n/r_n|} =. \frac{|a'_n|}{|a_n|} \frac{|1 + T'_n|}{|(1 + \Delta T_n)/(1 - r_n)|} \rightarrow 0 .$$

Our final two theorems are on infinite products.

THEOREM 23. *If the sequence* $\{1/a_n - 1/a_{n-1}\}$ *is bounded, then the complex product* $\Pi_0^\infty (1 + a_n)$ *diverges.*

Proof. Assume that $\Pi_0^\infty (1 + a_n)$ converges. Then $a_n \rightarrow 0$ and there is an $m \geqq 0$ such that for $k \geqq 0$, the quantities

$$S'_k = (1 + a_m)(1 + a_{m+1}) \cdots (1 + a_{m+k})$$

satisfy the limiting relation $S'_k \rightarrow S'$ for some $S' \neq 0$. We may assume that $m = 0$ so that $S'_n = \Pi_0^n (1 + a_i)$ for $n \geqq 0$. Since the sequence $\{(1 - r_n)/a_n\} = \{1/a_n - 1/a_{n-1}\}$ is bounded and $a_n \rightarrow 0$, we have $r_n \rightarrow 1$. Let $a'_0 = S'_0 = (1 + a_0)$ and $a'_n = S'_n - S'_{n-1} = \Pi_0^n (1 + a_i) - \Pi_0^{n-1} (1 + a_i) = [\Pi_0^{n-1} (1 + a_i)][(1 + a_n) - 1] = a_n \Pi_0^{n-1} (1 + a_i)$ for $n \geqq 1$. Then $1/a'_{n+1} - 1/a'_n = [1/[a_{n+1}(1 + a_n)] - 1/a_n]/\Pi_0^{n-1} (1 + a_i) = [(1/a_{n+1} - 1/a_n) - 1/[r_{n+1}(1 + a_n)]]/\Pi_0^{n-1} (1 + a_i)$. Hence, since $r_n \rightarrow 1$, $a_n \rightarrow 0$, $\{1/a_n - 1/a_{n-1}\}$ is bounded and $\Pi_0^\infty (1 + a_n) = S' \neq 0$, we see that $\{1/a'_{n+1} - 1/a'_n\}$ is bounded. From Tucker [2], $\Sigma a'_n$ diverges, i.e., $\Pi_0^\infty (1 + a_n)$ diverges.

THEOREM 24. *Suppose that* $|r_n| \leqq. \rho < 1$ *and* $a_n \neq -1$ *for all* n.

Then a necessary and sufficient condition for the δ^2-transform to accelerate the convergence of the infinite product $\Pi_0^\infty(1 + a_n)$ *is that* $\Delta r_n \to 0$.

Proof. Set $S'_n = \Pi_0^n(1 + a_i)$ for $n \geqq 0$, $a'_0 = S'_0$ and $a'_n = S'_n - S'_{n-1}$ for $n \geqq 1$. Since $|r_n| \leqq . \rho < 1$, we successively obtain the convergence of $\Sigma |a_n|$, $\Pi_0^\infty(1 + |a_i|)$ and $\Pi_0^\infty(1 + a_i) = S' = \Sigma a'_n \neq 0$. Also, $a_n \to 0$ and $r'_n = . r_n + a_n$ yield $|r'_n| \leqq . \rho' = (\rho + 1)/2 < 1$ and the equivalence of the conditions $\Delta r_n \to 0$ and $\Delta r'_n \to 0$. From Tucker [2], $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ if and only if $\Delta r'_n \to 0$. Hence, $\Sigma a'_{\delta n} \in MR(\Sigma a'_n)$ if and only if $\Delta r_n \to 0$.

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