$F'$-SPACES AND THEIR PRODUCT WITH $P$-SPACES

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The $F'$-spaces studied here, introduced by Leonard Gillman and Melvin Henriksen, are by definition completely regular Hausdorff spaces in which disjoint cozero-sets have disjoint closures. The principal result of this paper gives a sufficient condition that a product space be an $F'$-space and shows that the condition is, in a strong sense, best possible. A fortuitous corollary in the same vein responds to a question posed by Gillman: When is a product space basically disconnected (in the sense that each of its cozero-sets has open closure)?

A concept essential to the success of our investigation was suggested to us jointly by Anthony W. Hager and S. Mrowka in response to our search for a (simultaneous) generalization of the concepts “Lindelöf” and “separable.” Using the Hager-Mrowka terminology, which differs from that of Frolík in [3], we say that a space is weakly Lindelöf if each of its open covers admits a countable subfamily with dense union. §1 investigates $F'$-spaces which are (locally) weakly Lindelöf; §2 applies standard techniques to achieve a product theorem less successful than that of §3; §4 contains examples, chiefly elementary variants of examples from [5] or Kohls' [8], and some questions.

1. $F'$-spaces and their subspaces. Following [5], we say that a (completely regular Hausdorff) space is an $F'$-space provided that disjoint cozero-sets are completely separated (in the sense that some continuous real-valued function on the space assumes the value 0 on one of the sets and the value 1 on the other). It is clear that any $F'$-space is an $F'$-space and (by Urysohn’s Lemma) that the converse is valid for normal spaces. Since each element of the ring $C^*(X)$ of bounded real-valued continuous functions on $X$ extends continuously to the Stone-Čech compactification $\beta X$ of $X$, it follows that $X$ is an $F'$-space if and only if $\beta X$ is an $F'$-space. These and less elementary properties of $F'$-spaces are discussed at length in [5] and [6], to which the reader is referred also for definitions of unfamiliar concepts.

$F'$-spaces are characterized in 14.25 of [6] as those spaces in which each cozero-set is $C^*$-embedded. We begin with the analogous characterization of $F'$-spaces. All hypothesized spaces in this paper are understood to be completely regular Hausdorff spaces.

THEOREM 1.1. $X$ is an $F'$-space if and only if each cozero-set in $X$ is $C^*$-embedded in its own closure.
Proof. To show that coz $f$ (with $f \in C(X)$ and $f \geq 0$, say) is $C^*$-embedded in $\text{cl}_x \text{coz} f$ it suffices, according to Theorem 6.4 of [6], to show that disjoint zero-sets $A$ and $B$ in $\text{coz} f$ have disjoint closures in $\text{cl}_x \text{coz} f$. There exists $g \in C^*(\text{coz} f)$ with $g > 0$ on $A$, $g < 0$ on $B$. It is easily checked that the function $h$, defined on $X$ by the rule

$$h = \begin{cases} fg & \text{on } \text{coz} f \\ 0 & \text{on } Zf \end{cases}$$

lies in $C^*(X)$, and that the (disjoint) cozero-sets pos $h$, neg $h$, contain $A$ and $B$ respectively. Since $\text{cl}_x \text{pos} h \cap \text{cl}_x \text{neg} h = \emptyset$, we see that $A$ and $B$ have disjoint closures in $X$, hence surely in $\text{cl}_x \text{coz} f$.

The converse is trivial: If $U$ and $V$ are disjoint cozero-sets in $X$, then the characteristic function of $U$, considered as function on $U \cup V$, lies in $C^*(U \cup V)$, and its extension to a function in $C^*(\text{cl}_x(U \cup V))$ would have the values 0 and 1 simultaneously at any point in $\text{cl}_x(U \cap \text{cl}_x V)$.

The "weakly Lindelöf" concept described above allows us to show that certain subsets of $F'$-spaces are themselves $F'$, and that certain $F''$-spaces (for example, the separable ones) are in fact $F$-spaces. We begin by recording some simple facts about weakly Lindelöf spaces.

Recall that a subset $S$ of $X$ is said to be regularly closed if $S = \text{cl}_x \text{int}_x S$.

Lemma 1.2. (a) A regularly closed subset of a weakly Lindelöf space is weakly Lindelöf;

(b) A countable union of weakly Lindelöf subspaces of a (fixed) space is weakly Lindelöf;

(c) Each cozero-set in a weakly Lindelöf space is weakly Lindelöf.

Proof. (a) and (b) follow easily from the definition, and (c) is obvious since for $f \in C^*(X)$ the set $\text{coz} f$ is the union of the regularly closed sets $\text{cl}_x \{ x \in X : |f(x)| > 1/n \}$.

Lemma 1.2(c) shows that any point with a weakly Lindelöf neighborhood admits a fundamental system of weakly Lindelöf neighborhoods. For later use we formalize the concept with a definition.

Definition 1.3. The space $X$ is locally weakly Lindelöf at its point $x$ if $x$ admits a weakly Lindelöf neighborhood in $X$. A space locally weakly Lindelöf at each of its points is said to be locally weakly Lindelöf.

Theorem 1.4. Let $A$ and $B$ be weakly Lindelöf subsets of the
space $X$, each missing the closure (in $X$) of the other. Then there exist disjoint cozero-sets $U$ and $V$ for $X$ for which

$$A \subset \text{cl}_x(A \cap U), \quad B \subset \text{cl}_x(B \cap V).$$

**Proof.** For each $x \in A$ there exists $f_x \in C^*(X)$ with $f_x(x) = 0$, $f_x \equiv 1$ on $\text{cl}_x B$. Similarly, for each $y \in B$ there exists $g_y \in C^*(X)$ with $g_y(y) = 0$, $g_y \equiv 1$ on $\text{cl}_x A$. Taking $0 \leq f_x \leq 1$ and $0 \leq g_y \leq 1$ for each $x$ and $y$, we define

$$U_x = f_x^{-1}[0, 1/2], \quad V_y = g_y^{-1}[0, 1/2],$$

$$W_x = f_x^{-1}[0, 1/2], \quad Z_y = g_y^{-1}[0, 1/2].$$

Then, with $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ sequences chosen in $A$ and $B$ respectively so that $A \cap (\bigcup_n U_{x_n})$ is dense in $A$ and $B \cap (\bigcup_n V_{y_n})$ is dense in $B$, we set

$$U_\sim = U_{x_n} \setminus \bigcup_{k \leq n} Z_{y_k}, \quad V_\sim = V_{y_n} \setminus \bigcup_{k \leq n} W_{x_k}$$

and, finally, $U = \bigcup_n U_\sim$, $V = \bigcup_n V_\sim$.

The theorem just given has several elementary corollaries.

**Corollary 1.5.** Two weakly Lindelöf subsets of an $F'$-space, each missing the closure of the other, have disjoint closures (which are weakly Lindelöf).

**Corollary 1.6.** Any weakly Lindelöf subspace of an $F'$-space is itself an $F'$-space.

**Proof.** If $A$ and $B$ are disjoint cozero-sets in the weakly Lindelöf subset $Y$ of the $F'$-space $X$, we have from 1.2(c) that $A$ and $B$ are themselves weakly Lindelöf, and that

$$A \cap \text{cl}_Y B = A \cap \text{cl}_Y B = \emptyset \quad \text{and} \quad B \cap \text{cl}_X A = B \cap \text{cl}_X A = \emptyset.$$

From 1.5 it follows that

$$\emptyset = \text{cl}_X A \cap \text{cl}_X B \supset \text{cl}_Y A \cap \text{cl}_Y B.$$

**Corollary 1.7.** Each weakly Lindelöf subspace of an $F'$-space is $C^*$-embedded in its own closure.

**Proof.** Disjoint zero-sets of the weakly Lindelöf subspace $Y$ of the $F'$-space $X$ are contained in disjoint cozero subsets of $Y$, which by 1.2(c) and 1.5 have disjoint closures in $X$.

Corollaries 1.6 and 1.7 furnish us with a sufficient condition that an $F'$-space be an $F$-space.
THEOREM 1.8. Each $F'$-space with a dense Lindelöf subspace is an $F$-space.

Proof. If $Y$ is a dense Lindelöf subspace of the $F'$-space $X$, then $Y$ is $F'$ by 1.6, hence (being normal) is an $F$-space. But by 1.7 $Y$ is $C^*$-embedded in $X$, hence in $\beta X$, so that $\beta Y = \beta X$. Now $Y$ is an $F$-space, hence $\beta Y$, hence $\beta X$, hence $X$.

COROLLARY 1.9. A separable $F'$-space is an $F$-space.

The following simple result improves 3B.4 of [6]. Its proof, very similar to that of 1.4, is omitted.

THEOREM 1.10. Any two Lindelöf subsets of a (fixed) space, neither meeting the closure of the other, are contained in disjoint cozero-sets.

An example given in [5] shows that there exists a (nonnormal) $F'$-space which is not an $F$-space. For each such space $X$ the space $\beta X$, since it is normal, cannot be an $F'$-space; for (as we have observed earlier) $X$ is an $F$-space if and only if $\beta X$ is an $F$-space. Thus not every space in which an $F'$-space is dense and $C^*$-embedded need be an $F'$-space. The next result shows that passage to $C^*$-embedded subspaces is better behaved.

THEOREM 1.11. If $Y$ is a $C^*$-embedded subset of the $F'$-space $X$, then $Y$ is an $F'$-space.

Proof. Disjoint cozero-sets in $Y$ are contained in disjoint cozero-sets in $X$, whose closures (in $X$, even) are disjoint.

We shall show in Theorem 4.2 that the $F'$ property is inherited not only by $C^*$-embedded subsets, but by open subsets as well.

2. On the product of a (locally) weakly Lindelöf space and a $P$-space. A $P$-point in the space $X$ is a point $x$ with the property that each continuous real-valued function on $X$ is constant throughout some neighborhood of $x$. If each point of $X$ is a $P$-point, then $X$ is said to be a $P$-space. The $P$-spaces are precisely those spaces in which each $G_\delta$ subset is open.

The following diagram, a sub-graph of one found in [5] and in [8], is convenient for reference.

\[
\begin{array}{ccc}
\text{discrete} & \text{\textit{P}} & \text{basically disconnected} \\
\text{\textit{extremally disconnected}} & & \rightarrow F \rightarrow F'.
\end{array}
\]
In the interest of making this paper self-contained, we now include from [2] a proof of the fact that if a product space $X \times Y$ is an $F'$-space, then both $X$ and $Y$ are $F'$-spaces and either $X$ or $Y$ is a $P$-space. Indeed, the first conclusion is obvious. For the second, let $x_0$ and $y_0$ be points in $X$ and $Y$ respectively belonging to the boundary of the sets $\text{coz } f$ and $\text{coz } g$ respectively (with $f \in C(X)$ and $g \in C(Y)$ and $f \geq 0$ and $g \geq 0$). Then the function $h$, defined on $X \times Y$ by the rule $h(x, y) = f(x) - g(y)$, assumes both positive and negative values on each neighborhood in $X \times Y$ of $(x_0, y_0)$. Thus $\text{pos } h$ and $\text{neg } h$ are disjoint cozero-sets in $X \times Y$ each of whose closure contains $(x_0, y_0)$.

We are going to derive, in 2.4, a simple condition sufficient that a product space be an $F'$-space.

**Theorem 2.1.** Let $X$ be a $P$-space, let $Y$ be weakly Lindelöf, and let $f \in C^*(X \times Y)$. Then the real-valued function $F$, defined on $X$ by the rule

$$F(x) = \sup \{f(x, y) : y \in Y\},$$

lies in $C^*(X)$.

**Proof.** To check the continuity of $F$ at $x_0 \in X$, let $\varepsilon > 0$ and first find $y_0 \in Y$ such that $f(x_0, y_0) > F(x_0) - \varepsilon$. There is a neighborhood $U \times V$ of $(x_0, y_0)$ throughout which $f > F(x_0) - \varepsilon$, and for $x \in V$ we have $F(x) \geq f(x, y_0) > F(x_0) - \varepsilon$.

To find a neighborhood $U'$ of $x_0$ throughout which $F \leq F(x_0) + \varepsilon$, first select for each $y \in Y$ a neighborhood $U_y \times V_y$ of $(x_0, y)$ throughout which $f < F(x_0) + \varepsilon/2$. Because $Y$ is weakly Lindelöf there is a sequence $\{y_k\}_{k=1}^\infty$ in $Y$ with $\bigcup_k V_{y_k}$ dense in $Y$. With $U' = \bigcap_k U_{y_k}$ we check easily that $U'$ is a neighborhood of $x_0$ for which $F(x) \leq F(x_0) + \varepsilon$ whenever $x \in U'$.

**Corollary 2.2.** Let $X$ be a $P$-space and $Y$ a weakly Lindelöf space, and let $\pi$ denote the projection from $X \times Y$ onto $X$. Then for each cozero-set $A$ in $X \times Y$, the set $\pi A$ is open-and-closed in $X$.

**Proof.** If $A = \text{coz } f$ with $f \in C^*(X \times Y)$ and $f \geq 0$, then $\pi A$ is the cozero-set of the function $F$ defined as in 2.1, hence is closed (since $X$ is a $P$-space).

The following lemma asserts, in effect, that for suitably restricted spaces $X$ and $Y$, the closure in $X \times Y$ of each cozero-set may be computed by taking closures of vertical slices. When $A \subset X \times Y$ we denote $\text{cl}_{X \times Y} A$ by the symbol $\bar{A}$, and $A \cap \{x\} \times Y$ by $A_x$. 


Lemma 2.3. Let $X$ be a $P$-space and let $Y$ be locally weakly Lindelöf at each of its non-$P$-points. Then $\bar{A} = \bigcup_{x \in X} \bar{A}_x$ for each cozero-set $A$ in $X \times Y$.

Proof. The inclusion $\supseteq$ is obvious, so we choose $(x, y) \in \bar{A}$. We must show that $\{x\} \times V$ meets $A_x$ for each neighborhood $V$ in $Y$ of $y$. If $y$ is a $P$-point of $Y$ then $(x, y)$ is a $P$-point of $X \times Y$, so that indeed

$$(x, y) \in (\{x\} \times V) \cap A_x .$$

If $y$ is not a $P$-point of $Y$ and $V_0$ is a weakly Lindelöf neighborhood of $y$ in $Y$ with $V_0 \subseteq V$, then $(X \times V_0) \cap A$ is a cozero-set in $X \times V_0$ and 2.2 applies to yield: $\pi[(X \times V_0) \cap A]$ is open-and-closed in $X$. Since $(x, y) \in cl_{X \times V_0}(X \times V_0) \cap A$, we have

$$x = \pi(x, y) \in \pi cl_{X \times V_0}(X \times V_0) \cap A \subseteq cl_X \pi[(X \times V_0) \cap A] ,$$

so that $(\{x\} \times V) \cap A_x \supseteq (\{x\} \times V_0) \cap A_x \neq \emptyset$ as desired.

The elementary argument just given yields the following result, which we shall improve upon in 3.2.

Theorem 2.4. Let $Y$ be an $F'$-space which is locally weakly Lindelöf at each of its non-$P$-points. Then $X \times Y$ is an $F'$-space for each $P$-space $X$.

Proof. If $A$ and $B$ are disjoint cozero-sets in $X \times Y$, then from 2.3 we have

$$\bar{A} \cap \bar{B} = (\bigcup_{x \in X} \bar{A}_x) \cap (\bigcup_{x \in X} \bar{B}_x) = \bigcup_{x \in X} (\bar{A}_x \cap \bar{B}_x) = \bigcup_{x \in X} \emptyset = \emptyset .$$

The theorem just given furnishes a proof for 2.5(b) below, announced earlier in [2]. (In a letter of December 27, 1966, Professor Curtis has asserted his agreement with the authors’ beliefs that (a) the argument given in [2] contains a gap and (b) this error does not in any way affect the other interesting results of [2].)

Corollary 2.5. Let $X$ be a $P$-space and let $Y$ be an $F'$-space such that either

(a) $Y$ is locally Lindelöf; or

(b) $Y$ is locally separable.

Then $X \times Y$ is an $F'$-space.

Note added September 16, 1968. The reader may have observed already a fact noticed only lately by the authors: Each $F'$-space in which each open subset is weakly Lindelöf is extremally disconnected.
(in the sense that disjoint open subsets have disjoint closures). [For the proof, let $U$ and $V$ be disjoint open sets in such a space $Y$, suppose that $p \in \text{cl } U \cap \text{cl } V$, and for each point $y$ in $U$ find a cozero-set $U_y$ of $Y$ with $y \in U_y \subset U$. The cover $\{U_y: y \in U\}$ admits a countable subfamily $\mathcal{F}$ whose union is dense in $U$. If $\mathcal{F}$ is constructed similarly for $V$, then $\bigcup \mathcal{F}$ and $\bigcup \mathcal{F}'$ are disjoint cozero-sets in $X$ whose closures contain $p$.] It follows that each separable $F'$-space, and hence each locally separable $F'$-space, is extremally disconnected, and hence basically disconnected. Thus the conclusion to Corollary 2.5(b) is unnecessarily weak. In view of 3.4 we have in fact: If $X$ is a $P$-space and $Y$ is a locally separable $F'$-space, then $X \times Y$ is basically disconnected.

3. When the product of spaces is $F'$. It is clear that for each collection $\{W_\alpha\}_{\alpha \in A}$ of open covers of a locally weakly Lindelöf space $Y$ and for each $y$ in $Y$ one can find a neighborhood $U$ of $y$ and for each $\alpha$ a countable subfamily $\mathcal{V}_\alpha$ of $W_\alpha$ such that $U \subseteq \text{cl}_x(\bigcup \mathcal{V}_\alpha)$. (Indeed, the neighborhood $U$ may be chosen independent of the collection $\{W_\alpha\}_{\alpha \in A}$.)

When, in contrast to this strong condition, such a neighborhood $U$ is hypothesized to exist for each countable collection of covers of $Y$, we shall say that $Y$ is countably locally weakly Lindelöf (abbreviation: CLWL). The formal definition reads as follows:

**Definition 3.1.** The space $Y$ is CLWL if for each countable collection $\{W_n\}$ of open covers of $Y$ and for each $y$ in $Y$ there exist a neighborhood $U$ of $y$ and (for each $n$) a countable subfamily $\mathcal{V}_n$ of $W_n$ with $U \subseteq \text{cl}_x(\bigcup \mathcal{V}_n)$.

A crucial property of CLWL spaces is disclosed by the following lemma, upon which the results of this section depend.

For $f$ in $C(X \times Y)$, we denote by $f_x$ that (continuous) function on $Y$ defined by the rule $f_x(y) = f(x, y)$.

**Lemma 3.2.** Let $f \in C^*(X \times Y)$, where $X$ is a $P$-space and $Y$ is CLWL. If $(x_0, y_0) \in X \times Y$, then there is a neighborhood $U \times V$ of $(x_0, y_0)$ such that $f_x \equiv f_{x_0}$ on $V$ whenever $x \in U$.

**Proof.** For each $y$ in $Y$ and each positive integer $n$ there is a neighborhood $U_n(y) \times V_n(y)$ of $(x_0, y)$ for which

$$|f(x', y') - f(x_0, y)| < 1/n \quad \text{whenever } (x', y') \in U_n(y) \times V_n(y).$$

Since for each $n$ the family $\{V_n(y): y \in Y\}$ is an open cover of $Y$, there exist a neighborhood $V$ of $y_0$ and (for each $n$) a countable subset $Y_n$ of $Y$ for which $V \subseteq \text{cl}_x(\bigcup \{V_n(y): y \in Y_n\}).$
We define the neighborhood $U$ of $x_0$ by the rule

$$U = \bigcap_n (\cap \{U_n(y) : y \in Y_n\}).$$

To check that neighborhood $U \times V$ of $(x_0, y_0)$ is as desired, suppose that there is a point $(x', y')$ in $U \times V$ with $f(x', y') \neq f(x_0, y_0)$. Choosing an integer $n$ and a neighborhood $U' \times V'$ of $(x', y')$ such that $|f(x, y) - f(x_0, y')| > 1/n$ whenever $(x, y) \in U' \times V'$, we see that since $y' \in V \subset \text{cl}_r(\cup \{V_{3n}(y) : y \in Y_{3n}\})$ and $V' \cap V_{3n}(y')$ is a neighborhood of $y'$ there exist points $y$ in $Y_{3n}$ and $y'$ in $[V' \cap V_{3n}(y')] \cap V_{3n}(y')$.

Since $(x', y') \in U' \times V'$, we have

$$|f(x', y') - f(x_0, y')| > 1/n.$$

But since $(x', y) \in U \times V_{3n}(y) \subset U_{3n}(y) \times V_{3n}(y)$, and $(x_0, y) \in U_{3n}(y) \times V_{3n}(y)$, and $(x_0, y) \in U_{3n}(y) \times V_{3n}(y')$, we have

$$|f(x', y) - f(x_0, y')| \leq |f(x', y') - f(x_0, y')|$$

$$+ |f(x_0, y') - f(x_0, y)| + |f(x_0, y) - f(x_0, y')|$$

$$< 1/3n + 1/3n + 1/3n = 1/n.$$

We have seen in § 2 that if the product space $X \times Y$ is an $F'$-space then both $X$ and $Y$ are $F''$-spaces and either $X$ or $Y$ is a $P$-space. It is clear that every discrete space is a $P$-space, and that the product of any $F'$-space with a discrete space is an $F''$-space; the example given by Gillman in [4], however, shows that the product of a $P$-space with an $F'$-space may fail to be an $F''$-space. Thus it appears natural to ask the question: Which $F'$-spaces have the property that their product with each $P$-space is an $F''$-space? We now answer this question.

**Theorem 3.3.** In order that $X \times Y$ be an $F'$-space for each $P$-space $X$, it is necessary and sufficient that $Y$ be an $F'$-space which is CLWL.

**Proof.** Sufficiency. Let $f \in C^*(X \times Y)$, and let $(x_0, y_0) \in X \times Y$. We may suppose without loss of generality that there is a neighborhood $V'$ of $y_0$ in $Y$ for which

$$V' \cap \text{pos } f_{x_0} = \emptyset.$$

But then, choosing $U \times V$ as in Lemma 3.2, we see that

$$U \times (V \cap V') \cap \text{pos } f = \emptyset,$$

so that $(x_0, y_0) \in \text{cl } \text{pos } f$.

Necessity. (A preliminary version of the construction below—in the context of weakly Lindelöf spaces, not of CLWL spaces—was
communicated to us by Anthony W. Hager in connection with a project not closely related to that of the present paper. We appreciate professor Hager’s helpful letter, which itself profited from his collaboration with S. Mrowka.)

We have already seen that $Y$ must be an $F'$-space. If $Y$ is not CLWL then there are a sequence $\{\mathcal{W}_n\}$ of open covers of $Y$ and a point $y_0$ in $Y$ with the property that for each neighborhood $U$ of $y_0$ there is an integer $n(U)$ for which the relation $U \subset \text{cl}_{f'}(\bigcup \mathcal{W}_n)$ fails for each countable subfamily of $\mathcal{W}_n$. Let $\mathcal{W}$ denote the collection of neighborhoods of $y_0$. With each $U \in \mathcal{W}$ we associate the family $\Sigma(U)$ of countable intersections of sets of the form $Y \setminus W$ with $W \in \mathcal{W}_n(U)$, and we write

$$\tau(U) = \{(A, U) : A \in \Sigma(U)\}.$$ 

From the definition of $n(U)$ it follows that $(\text{int}_{Y} A) \cap U \neq \emptyset$ whenever $A \in \Sigma(U)$. The space $X$ is the set $\{\infty\} \cup \bigcup_{U \in \mathcal{W}} \tau(U)$, topologized as follows: Each of the points $(A, U)$, for $A \in \Sigma(U)$, constitutes an open set, so that $X$ is discrete at each of its points except for $\infty$; and a set containing the point $\infty$ is a neighborhood of $\infty$ if and only if it contains, for each $U \in \mathcal{W}$, some point $(A, U) \in \tau(U)$ and each point of the form $(B, U)$ with $B \subseteq A$ and $(B, U) \in \tau(U)$. Since $\bigcap_{k=1}^n A_k \in \Sigma(U)$ whenever each $A_k \in \Sigma(U)$, it follows that each countable intersection of neighborhoods of $\infty$ is a neighborhood of $\infty$, so that $X$ is a $P'$-space. Like every Hausdorff space with a basis of open-and-closed sets, $X$ is completely regular. It remains to show that $X \times Y$ is not an $F'$-space.

Since for $U \in \mathcal{W}$ there is no countable subfamily $\mathcal{W}'$ of $\mathcal{W}_{n(U)}$ for which $U \subset \text{cl}_{f'}(\bigcup \mathcal{W}')$, the set $(\text{int}_{Y} A) \cap U$ is uncountable whenever $U \in \mathcal{W}$ and $A \in \Sigma(U)$. Thus whenever $(A, U) \in \tau(U)$ we choose distinct points $p_{(A, U)}$ and $q_{(A, U)}$ in $(\text{int}_{Y} A) \cap U$ and disjoint neighborhoods $F_{(A, U)}$ of $p_{(A, U)}$ and $G_{(A, U)}$ of $q_{(A, U)}$ respectively, with $F_{(A, U)} \cup G_{(A, U)} \subseteq (\text{int}_{Y} A) \cap U$. Because $Y$ is completely regular there exist continuous functions $f_{(A, U)}$ and $g_{(A, U)}$ mapping $Y$ into $[0, 1]$ such that

$$f_{(A, U)}(p_{(A, U)}) = 1, \quad f_{(A, U)} \equiv 0 \text{ off } F_{(A, U)},$$

$$g_{(A, U)}(q_{(A, U)}) = 1, \quad g_{(A, U)} \equiv 0 \text{ off } G_{(A, U)}.$$ 

Now for each positive integer $k$ we define functions $f_k$ and $g_k$ on $X \times Y$ by the rules $f_k(x, y) = g_k(x, y) = 0$ if $x = \infty$ or if $x = (A, U)$ with $k \neq n(U)$; $f_k((A, U), y) = f_{(A, U)}(y)$ if $k = n(U)$; $g_k((A, U), y) = g_{(A, U)}(y)$ if $k = n(U)$. Each function $f_k$ is continuous at each point $((A, U), y) = (x, y) \in X \times Y$ (with $x \neq \infty$), since $f_k$ agrees either with the function $0$ or with the continuous function $f_{(A, U)} \circ \pi_Y$ on the open
subset \( \{(A, U)\} \times Y \) of \( X \times Y \). Similarly, each function \( g_k \) is continuous at each point \((x, y) \in X \times Y\) with \( x \neq \infty \). To check the continuity (of \( f_k \), say) at the point \((\infty, y) \in X \times Y\), find \( W \in \mathcal{W}_x \) for which \( y \in W \) and write

\[
V = \{\infty\} \cup \bigcup_{k \neq n(U)} \tau(U) \cup \bigcup_{k = n(U)} \{(B, U) : B \subset Y \setminus W\}.
\]

Then \( V \times W \) is a neighborhood of \((\infty, y)\) on which \( f_k \) is identically 0: For if \((A, U) \in \tau(U)\) with \( k \neq n(U)\) we have \( f_k((A, U), y) = 0 \), and if \( A \in \Sigma(U) \) with \( A \subset Y \setminus W \) and \( k = n(U) \), then (since \( y \in W \subset Y \setminus \text{int}, A \subset Y \setminus F(A, U)\)) we have

\[
f_k((A, U), y) = f_{(A, U)}(y) = 0.
\]

We notice next that if \( k \) and \( m \) are positive integers then \( \text{coz} f_k \cap \text{coz} g_m = \emptyset \): Indeed, if \( f_k((A, U), y) \neq 0 \) and \( g_m((A, U), y) \neq 0 \), then \( k = n(U) \) and \( m = n(U) \), so that \( y \in F(A, U) \cap G(A, U) \), a contradiction. Thus, defining

\[
f = \sum_{k=1}^{\infty} f_k/2^k \quad \text{and} \quad g = \sum_{k=1}^{\infty} g_k/2^k
\]

we have \( f \in C^*(X \times Y) \) and \( g \in C^*(X \times Y) \) and \( \text{coz} f \cap \text{coz} g = \emptyset \). Nevertheless for each neighborhood \( V \times U_0 \) of \((\infty, y_0)\) we have \((A_0, U_0) \in V\) for some \( A_0 \in \Sigma(U_0) \), so that

\[
f((A_0, U_0), p_{(A_0, U_0)}) \leq f_{n(U_0)}((A_0, U_0), p_{(A_0, U_0)})/2^{n(U_0)}
\]

\[
= f_{(A_0, U_0)}(p_{(A_0, U_0)})/2^{n(U_0)} = 1/2^{n(U_0)} > 0
\]

and \((V \times U_0) \cap \text{coz} f \neq \emptyset\). Likewise \((V \times U_0) \cap \text{coz} g \neq \emptyset\), and it follows that \((\infty, y_0) \in \text{cl coz} f \cap \text{cl coz} g \). Thus \( X \times Y \) is not an \( F'\)-space.

The proof of Theorem 3.3 being now complete, we turn to the corollary which we believe responds adequately to Gillman’s request in [4] for a theorem characterizing those pairs of spaces \((X, Y)\) for which \( X \times Y \) is basically disconnected.

**Corollary 3.4.** In order that \( X \times Y \) be basically disconnected for each \( P\)-space \( X \), it is necessary and sufficient that \( Y \) be a basically disconnected space which is \( \text{CLWL} \).

**Proof.** Sufficiency. Let \((x_0, y_0) \in \text{cl coz} f \), where \( f \in C^*(X \times Y) \), and let \( V' \) be a neighborhood of \( y_0 \) in \( Y \) for which \( V' \subset \text{cl coz} f_{x_0} \). Choosing \( U \times V \) as in Lemma 3.2, we see that \( U \times (V \cap V') \) is a neighborhood in \( X \times Y \) of \((x_0, y_0)\) for which

\[
U \times (V \cap V') \subset \text{cl coz} f.
\]
Necessity. That $Y$ must be basically disconnected is clear. That $Y$ must be CLWL follows from 3.3 and the fact that each basically disconnected space is an $F'$-space.

4. Some examples and questions. If the point $x$ of the topological space $X$ admits a neighborhood ($X$ itself, say) which is an $F'$-space, then each neighborhood $U$ of $x$ in $X$ contains a neighborhood $V$ which is an $F'$-space: Indeed, if $f \in C(X)$ with $x \in \text{coz } f \subset U$ and we set $V = \text{coz } f$, then each pair $(A, B)$ of disjoint cozero-sets of $V$ is a pair of disjoint cozero-sets in $X$, which accordingly may be completely separated in $X$, hence in $V$.

The paragraph above shows that any point with a neighborhood which is an $F'$-space admits a fundamental system of $F'$-space neighborhoods. The statement with “$F'$" replaced throughout by “$F''$" follows from the implication (b) $\Rightarrow$ (d) of Theorem 4.2 below. The following definitions are natural.

**Definition 4.1.** The space $X$ is locally $F$ (resp. locally $F''$) at the point $x \in X$ if $x$ admits a neighborhood in $X$ which is an $F$-space (resp. an $F''$-space).

Clearly each $F$-space is locally $F'$, and each locally $F'$ space is locally $F''$. Gillman and Henriksen produce in 8.14 of [5] an $F'$-space which is not an $F$-space, and their space is easily checked to be locally $F'$. In the same spirit we shall present in 4.3 an $F''$-space which is not locally $F'$. We want first to make precise the assertion that the $F'$ property, unlike the $F$ property, is a local property.

**Theorem 4.2.** For each space $X$, the following properties are equivalent:

(a) $X$ is an $F'$-space;
(b) $X$ is locally $F'$;
(c) each cozero-set in $X$ is an $F'$-space;
(d) each open subset of $X$ is an $F'$-space.

**Proof.** That (a) $\Rightarrow$ (b) is clear. To see that (b) $\Rightarrow$ (c), let $U$ be a cozero-set in $X$ and let $A$ and $B$ be disjoint (relative) cozero subsets of $U$. Then $A$ and $B$ are disjoint cozero subsets of $X$. Suppose $p \in \text{cl}_v A \cap \text{cl}_v B$. Then, if $V$ is the hypothesized $F'$-space neighborhood of $p$, we have $p \in \text{cl}_v (A \cap V) \cap \text{cl}_v (B \cap V)$. This contradicts the fact that $V$ is an $F'$-space.

If (c) holds and $A$ and $B$ are disjoint (relative) cozero-sets of an open subset $U$ of $X$, then for any point $p$ in $\text{cl}_v A \cap \text{cl}_v B$ there exists a cozero-set $V$ in $X$ for which $p \in V \subset U$. It follows that $p \in \text{cl}_v (A \cap V) \cap \text{cl}_v (B \cap V)$,
contradicting the fact that $V$ is an $F'$-space. This contradiction shows that (d) holds.

The implication $(d) \Rightarrow (a)$ is trivial.

**Example 4.3.** An $F'$-space not locally $F$. Let $X$ be any $F'$-space which is not an $F$-space, let $D$ be the discrete space with $|D| = \aleph_1$, and let $Y = (X \times D) \cup \{\infty\}$, where $\infty$ is any point not in $X \times D$ and $Y$ is topologized as follows: A subset of $X \times D$ is open in $Y$ if it is open in the usual product topology on $X \times D$, and $\infty$ has an open neighborhood basis consisting of all sets of the form $\{\infty\} \cup (X \times E)$ with $|D \setminus E| \leq \aleph_0$. Then $\infty$ admits no neighborhood which is an $F$-space, since each neighborhood of $\infty$ contains (for some $d \in D$) the set $X \times \{d\}$, which is homeomorphic to $X$ itself, as an open-and-closed subset. Yet $Y$ is an $F'$-space since $\infty$ is a $P$-point of $Y$ and each other point of $Y$ belongs to an $F'$-space, $X \times D$, which is dense in $Y$.

We have observed already that a Lindelöf $F'$-space, being normal, is an $F$-space. We show next that the Lindelöf condition cannot be replaced by the locally Lindelöf property.

**Example 4.4.** A locally Lindelöf $F'$-space which is not $F$. The space $X = L' \times L \setminus \{\omega_2, \omega_3\} \cup \bigcup_{\alpha < \omega_1} D_\alpha$ defined in 8.14 of [5] does not fill the bill here because the space $L'$ of ordinals $\leq \omega_5$ (with each $\gamma < \omega_2$ isolated and with neighborhoods of $\omega_5$ as in the order topology) is not Lindelöf. When the space is modified by the replacement of $L'$ by $\beta L'$, the resulting space ($X'$ say) fails to be an $F$-space just as in [5]. Yet $L'$ is a $P$-space, so that $\beta L'$ is a compact $F$-space, and therefore (by Theorem 3.3 above, or by Theorem 6.1 of [9]) $\beta L' \times L$ is a Lindelöf $F'$-space. Thus $X'$ is a locally Lindelöf space which is locally $F'$, hence is a locally Lindelöf $F'$-space.

The condition that a space be locally weakly Lindelöf at each of its non-$P$-points is more easily worked with than the condition that it be CLWL. A converse to Theorem 2.4 would, therefore, be a welcome replacement for the “necessity” part of Theorem 3.3. The following example shows that the converse to Theorem 2.4 is invalid.

**Example 4.5.** A CLWL $F'$-space with a non-$P$-point at which it is not locally weakly Lindelöf. Let $Y$ be the space $D \times D \cup \{\infty\}$ with $D$ the discrete space for which $|D| = \aleph_1$ and (after the fashion of 8.5 of [5]) adjoin to $Y$ a copy of the integers $N$ so that $\infty$ becomes a point in $\beta N \setminus N$. The resulting space $Y' = Y \cup N$ is topologized so that each point $y \neq \infty$ constitutes by itself an open set, while a set containing $\infty$ is a neighborhood of $\infty$ if it contains both a set drawn from the ultra-
filter on \( N \) corresponding to \( \infty \) and a set of the form \( D \times E \) with \( |D \times E| \leq \aleph_0 \). Then \( \infty \) is not a \( P \)-point of \( Y' \), since the function whose value at the integer \( n \in N \subseteq Y' \) is \( 1/n \) and whose value at each other point of \( Y' \) is 0 is constant on no neighborhood of \( \infty \); and \( Y' \) is not locally weakly Lindelöf at \( \infty \) since each neighborhood of \( \infty \) contains as an open-and-closed subset a homeomorph of the uncountable discrete space \( D \). The only nonisolated point of \( Y' \), \( \infty \), can belong to a set of the form \( (\text{cl} \; \text{co} \; f) \setminus \text{co} \; f \) only when \( \infty \in \text{cl}(\text{co} \; f \cap N) \), so that \( Y' \) is an \( F' \)-space. If, finally, \( \mathcal{V}^\infty \) is a sequence of open covers of \( Y' \) and a neighborhood \( U \) of \( \infty \) in \( Y \) is chosen so that for each \( n \) we have \( U \subseteq W_n \) for some \( W_n \in \mathcal{V}^\infty \) (as is possible, since \( Y \) is a \( P \)-space), then evidently \( U \cup N \) is a neighborhood of \( \infty \) in \( Y' \) contained in \( \text{cl}_F (U \cup \mathcal{V}^\infty) \) for a suitable countable subfamily \( \mathcal{V}_n \) of \( \mathcal{V}^\infty \). Thus \( Y' \) is \( \text{CLWL} \).

Theorem 1.8 does not provide an answer to the following problem, which we have been unable to solve.

**QUESTION 4.6.** Is each weakly Lindelöf \( F' \)-space an \( F \)-space?

On the basis of Theorem 3.3 and Corollary 3.4 and the fact that the class of \( F \)-spaces is nestled properly between the classes of \( F' \)- and of basically disconnected spaces, one wonders whether the obvious \( F \)-space analogue of 3.3 and 3.4 is true. We have not been able to settle this question, though one of us hopes to pursue it in a later communication. We close with a formal statement of this question, and of a related problem.

**QUESTION 4.7.** In order that \( X \times Y \) be an \( F \)-space for each \( P \)-space \( X \), is it sufficient that \( Y \) be an \( F \)-space which is \( \text{CLWL} \)?

**QUESTION 4.8.** Do there exist a \( P \)-space \( X \) and an \( F \)-space \( Y \) such that \( X \times Y \) is an \( F' \)-space but not an \( F \)-space?

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